
by JD Clayton


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A continuum theory of the mechanical behavior of solid materials is presented wherein fundamental geometric quantities such as the metric tensor and connection coefficients can depend on one or more director vectors, also called internal state vectors. This theory, referred to as generalized pseudo-Finsler geometric continuum mechanics, enables depiction of a very broad class of physical phenomena in deformable solid bodies. The general nonlinear theory is reported first, primarily summarizing prior work by the author. Next, a new application of the theory to torsional deformation of solids is presented, whereby a cylindrical sample of material may be simultaneously subjected to twisting, extension or compression along its axis, as well as possible radial confinement or contraction. The internal state variable, when normalized by a regularization length, is identified with an order parameter associated with inelastic deformation that may include slippage, fracture, and/or structural collapse. Evolution of the internal state follows a generalized Ginzburg–Landau type of kinetic equation. For axially homogeneous fields, a coupled system of nonlinear partial differential equations is obtained that can be integrated numerically. Results are first documented for generic solids representative of many crystals that exist in nature. Then solutions corresponding to realistic properties of crystals of boron carbide ceramic and ice are reported. Results for boron carbide predict a dominant effect of shearing over compression on the structural transformation process, in agreement with observations from atomistic simulations. Results for ice demonstrate periods of steady plastic flow under constant applied average shear stress as well as torsional rigidity varying with sample size. Existence of both phenomena agrees with experimental observations.

**Subject Terms**
differential geometry, phase field, nonlinear elasticity, continuum mechanics, microstructure

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**Abstract**
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Generalized pseudo-Finsler geometry applied
to the nonlinear mechanics of torsion
of crystalline solids

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A continuum theory of the mechanical behavior of solid materials is presented wherein fundamental geometric quantities such as the metric tensor and connection coefficients can depend on one or more director vectors, also called internal state vectors. This theory, referred to as generalized pseudo-Finsler geometric continuum mechanics, enables depiction of a very broad class of physical phenomena in deformable solid bodies. The general nonlinear theory is reported first, primarily summarizing prior work by the author. Next, a new application of the theory to torsional deformation of solids is presented, whereby a cylindrical sample of material may be simultaneously subjected to twisting, extension or compression along its axis, as well as possible radial confinement or contraction. The internal state variable, when normalized by a regularization length, is identified with an order parameter associated with inelastic deformation that may include slippage, fracture, and/or structural collapse. Evolution of the internal state follows a generalized Ginzburg–Landau type of kinetic equation. For axially homogeneous fields, a coupled system of nonlinear partial differential equations is obtained that can be integrated numerically. Results are first documented for generic solids representative of many crystals that exist in nature. Then solutions corresponding to realistic properties of crystals of boron carbide ceramic and ice are reported. Results for boron carbide predict a dominant effect of shearing over compression on the structural transformation process, in agreement with observations from atomistic simulations. Results for ice demonstrate periods of steady plastic flow under constant applied average shear stress as well as torsional rigidity varying with sample size. Existence of both phenomena agrees with experimental observations.

Keywords: Differential geometry; phase field; nonlinear elasticity.

Mathematics Subject Classification 2010: 53Z05, 74A45, 74A60, 74B20
1. Introduction

Theoretical work in geometric mechanics of crystalline substances, i.e. single crystals and polycrystals, dates to the mid-20th century. Many early works [1, 2] focused on crystals with dislocations, the primary carriers of plastic deformation in these kinds of solids [3, 4]. Differential geometry continues to be prominently featured in modern treatments of continuum physics of solids with dislocations [5, 6] as well as those with other kinds of lattice defects such as disclinations [4, 7], point defects [8, 9], fractures [10], and deformation twins [3, 12]. Sophisticated geometric methods have also been forwarded for analysis of thermomechanics coupled with electromagnetism in oriented solids [13, 14]. Importantly in the present context, Amari [15] introduced the use of Finsler geometry in solid mechanics in a pioneering theory of ferromagnetic crystals undergoing possibly large elastic and plastic deformations. Other early propositions of Finsler geometry to applications in continuum physics of solids were outlined by Kondo [16] and Eringen [17].

In this paper, a novel theory incorporating some concepts from a generalization of pseudo-Finsler geometry [18] is used to address the physics of irreversible deformations and structural changes in solids under finite deformation, flow, and/or fracture. These deformations and structural changes may most generally emerge from dislocations, disclinations, crack opening or sliding, shear bands, voids, pore collapse, or solid-solid phase changes. This paper advances prior work by the author summarized in the following. In [19], the formal mathematical theory was first presented in the context of nonlinear elasticity of solids with microstructure, building on findings presented in a literature review [20] and preliminary report [21]. The geometric-kinematic framework was extended in [22, 23] to explicitly include inelastic deformations manifesting as anholonomic contributions [24, 25] to the deformation gradient, i.e. the tangent map between referential and spatial configurations of the body manifold. Works cited above all considered the quasi-static case and the correspondingly derived Euler–Lagrange equations for static equilibrium. Governing equations for dynamic case were derived, with particular analysis focused on planar shock waves, in [26]. Numerical solutions for polycrystals undergoing structural changes were recently reported in [27].

This work extends and newly applies the dynamic theory of generalized Finsler-geometric continuum mechanics to torsional deformation. A cylindrical sample is twisted and perhaps simultaneously compressed or stretched along its axis of symmetry. Radial compression or confinement may also take place along the lateral surface of the cylinder. Such work has been of keen interest to the physics and mechanics communities since the pioneering studies of Bridgman [28] that contributed to his winning of the Nobel Prize in Physics in 1946. A recent review of developments on the topic of high-pressure torsion experiments in the 20th century is given in [29]. Modern research often involves rotational diamond anvil cells that enable study of various structural transformations under simultaneous extremely high pressure and high shearing stress.
A brief summary of prior models of this class of problems is in order. Non-linear elastic treatments of cylinders undergoing torsion with possible axial extension/compression exist, but exact analytical solutions are scarce for compressible bodies \[30\] with the exception of those described by special constitutive laws \[31\]. In general compressible solids, numerical methods are needed to solve the radial momentum balance. For cases involving plastic flow and phase changes under simultaneous compression and torsion, numerical modeling \[32, 33\] incorporating the finite element method has provided insight that complements rotational diamond anvil cell experiments. For pure torsion of metallic wires at the micron scale, size effects (e.g. smaller is stronger) have been modeled in the context of a simplified geometrically linear deformation theory of elasto-plasticity \[34\]. Size effects in torsion were analyzed quantitatively via an earlier appearing gradient plasticity theory \[35, 36\] in \[37–39\]. The usual Hall–Petch relation and its inverse behavior were modeled via a gradient theory of plasticity with surface tension in \[40\] and in several more recent works \[41, 42\]. Mechanics of single crystal rods has also been studied analytically within the framework of continuum elastic–plastic dislocation theory \[43\].

Nonlinear theories of materials with microstructure were advanced in several other relevant works with a basis in differential geometry appearing in the late 20th century \[44–48\]. A theory invoking a multiplicative decomposition of the deformation gradient, the geometrically defined dislocation density tensor, and the dislocation drift rate was established in \[45\]. An extensive treatment of governing and constitutive equations, and their link with engineering plasticity, was given in \[46\]. A solution of the problem of determination of the anholonomic crystal reference configuration (i.e. an incompatible “intermediate configuration”) given the elastic strain metric and torsion tensor (i.e. dislocation density tensor) was obtained in \[47\]. A comprehensive treatment of aforementioned topics in the context of elastic–plastic crystals, along with mathematical preliminaries as well as geometric linearization, is given in the monograph \[44\]. A more general theoretical approach for oriented media, including geometry, kinematics, and governing mechanics equations, is reported in \[48\]. This theory \[48\], which can reproduce the elastic–plastic crystal description under certain physical assumptions, features a set of director vectors attached to each material point, similarly to \[49\] and the model used in this work. A fundamental distinction is that the model of the present work invokes concepts from generalized Finsler geometry.

This paper provides a new contribution by considering simultaneous finite inelastic deformations in the shearing/twisting direction (as would occur for pure torsion) and the axial and radial directions. The former isochoric mode physically may correspond to dislocation glide, twinning, slip bands, or fracture and subsequent sliding of failed surfaces. The latter volumetric mode may physically encompass strain contributions from displacive phase transformations, dilatation from voids or cracks, or contraction from pore collapse. In the context of the theory, an
order parameter associated with thermodynamically irreversible changes in internal state — which geometrically enters the state vector of generalized pseudo-Finsler space — is mathematically linked to these mechanism(s). For isothermal processes, the free energy density stored in the solid depends on elastic deformation, the order parameter, and the order parameter’s horizontal covariant derivative. The latter two dependencies enable consideration of elastic softening and surface energy of failed zones similarly to phase field models [50]. As introduced in [26], a kinetic equation for evolution of internal state is developed from the residual departure from equilibrium derived in [19]. For homogeneous axial and torsional straining, the governing equations reduce to a system of coupled nonlinear partial differential equations in terms of the radial coordinate and time that are solved numerically. Solutions are then studied in the context of several real crystalline solids, specifically ice single crystals [51] and boron carbide ceramic [52, 53]. To obtain predictive results, the model, with a few exceptions, requires only fundamental measurable material properties, without extensive fitting or calibration.

The paper is structured as follows. A general theory applicable to pseudo-Finsler geometric spaces of dimension three is given in Sec. 2. This includes documentation of geometric definitions, kinematic assumptions, thermodynamics, and balance laws. Application to the torsion problem is undertaken in Sec. 3, ultimately leading to a system of partial differential equations dictating mechanical equilibrium and order parameter kinetics. Solutions and insights obtained are reported in Sec. 4, followed by concluding remarks in Sec. 5.

2. Differential-Geometric Theory: Governing Equations

The general theory applicable to description of a three-dimensional solid body subjected to arbitrary boundary conditions, with a few caveats discussed next, is presented here in Sec. 2. Loading conditions are limited to quasi-static mechanics, such that static conservation laws of linear and angular momentum apply. These laws are sufficient for modeling torsion–compression experiments later in Secs. 3 and 4, for example, which do not involve acoustic wave propagation. Dynamic evolution of internal state is enabled leading to realistic descriptions of dissipative processes such as viscoplasticity and dynamic fracture. The presentation of Sec. 2 closely follows prior works [22, 23] wherein multiplicative kinematics were introduced, leading to Euler–Lagrange equations of equilibrium. Detailed derivations or proofs of some relations can also be found in [19].

2.1. Generalized pseudo-Finsler geometry

The Finsler-geometric representations of the reference and spatial configurations of a material manifold are presented in Secs. 2.1.1 and 2.1.2. Tangent mappings between these configurations are presented in Sec. 2.1.3. Multiplicative kinematics distinguishing between elastic and inelastic deformations are discussed in Sec. 2.1.4.
2.1.1. Reference configuration geometry

Let $\mathfrak{M}$ be a differential manifold of spatial dimension 3. This manifold is physically identified with a material body embedded in ambient Euclidean 3-space. Let $X \in \mathfrak{M}$ denote a material point, and let $\{X^A\} (A = 1, 2, 3)$ denote a coordinate chart covering part or all of $\mathfrak{M}$. Each point is assigned a vector $D$ that serves as a descriptor of material microstructure. Coordinates $\{D^A\} (A = 1, 2, 3)$ are the entries of $D$; this set of coordinates can be associated with a chart on a second manifold $\mathfrak{U}$, also of dimension 3. The vector $D$, referred to here as a state vector or internal state vector, need not be of unit length. Regarding notation, dependence of a function on $(X, D)$ implies dependence on reference charts $\{(X^A), \{D^A\}\}$. Occasionally the notation $X$ is used for a position vector on the reference chart.

Similar to the theory reported in [54], the reference state is now described in terms of a generalized pseudo-Finsler geometry. Let $Z = (\mathfrak{Z}, \Pi, \mathfrak{M}, \mathfrak{U})$ be a fiber bundle of total space $\mathfrak{Z}$ (dimension 6), where $\Pi : \mathfrak{Z} \to \mathfrak{M}$ is the projection and $\mathfrak{U}$ the fiber. A coordinate chart on $\mathfrak{Z}$ is the set $\{X, D\}$. Holonomic basis vectors on $\mathfrak{Z}$ are the fields $\{\frac{\partial}{\partial X^A}, \frac{\partial}{\partial D^A}\}$. Let $N^A_B(X, D)$ denote nonlinear connection coefficients as are standardly introduced in Finsler geometry. A non-holonomic basis whose entries transform between coordinate systems as typical vectors when $N^A_B$ have certain properties [54] is

$$\frac{\delta}{\delta X^A} = \frac{\partial}{\partial X^A} - N^B_A \frac{\partial}{\partial D^B}, \quad \delta D^A = dD^A + N^A_B dX^B. \tag{1}$$

The set $\{\frac{\delta}{\delta X^A}, \frac{\partial}{\partial D^A}\}$ is used for a local basis on the tangent bundle $T\mathfrak{Z}$, and the reciprocal set $\{dX^A, \delta D^A\}$ for the cotangent bundle $T^*\mathfrak{Z}$. The Sasaki metric tensor is the symmetric entity

$$G(X, D) = G_{AB}(X, D) dX^A \otimes dX^B + G_{AB}(X, D) \delta D^A \otimes \delta D^B. \tag{2}$$

Components $G_{AB}$ ($G^{AB}$) are used to lower (raise) indices as in tensor calculus. The determinant of the covariant metric is written as $G(X, D) = \det[G_{AB}(X, D)]$.

Partial coordinate differentiation and delta-differentiation are denoted compactly as

$$\partial_A() = \frac{\partial()}{\partial X^A}, \quad \bar{\partial}_A() = \frac{\partial()}{\partial D^A}, \quad \delta_A() = \frac{\delta()}{\delta X^A} = \partial_A() - N^B_A \bar{\partial}_B(). \tag{3}$$

The Christoffel symbols of the second kind for the Levi-Civita connection on $\mathfrak{M}$ are

$$\gamma^A_{BC} = \frac{1}{2} G^{AD}(\partial_C G_{BD} + \partial_B G_{CD} - \partial_D G_{BC}) = G^{AD} \gamma_{BCD}. \tag{4}$$

Cartan’s tensor referred to material space is

$$C^A_{BC} = \frac{1}{2} G^{AD}(\partial_D G_{BD} + \partial_B G_{CD} - \partial_C G_{BD}) = G^{AD} C_{BCD}. \tag{5}$$

The horizontal coefficients of the Chern–Rund and Cartan connections [54] [55] [18] are the following equivalent quantities:

$$\Gamma^A_{BC} = \frac{1}{2} G^{AD}(\delta_C G_{BD} + \delta_B G_{CD} - \delta_D G_{BC}) = G^{AD} \Gamma_{BCD}. \tag{6}$$
The spray and canonical nonlinear connection coefficients derived from it, the latter an example of those in [11] when $N_B^A = G_B^A$, are

$$G^A = \frac{1}{2} \gamma_{BC}^A D^B D^C, \quad G_B^A = \partial_B G^A. \tag{7}$$

Let $\nabla(\cdot)$ denote a covariant derivative. Horizontal gradients of basis vectors are

$$\nabla_{/X^{\mu}} \frac{\delta}{\delta X^{\nu}} = H_{BC}^A \frac{\delta}{\delta X^A}, \quad \nabla_{/X^{\mu}} \frac{\partial}{\partial D^A} = K_{BC}^A \frac{\partial}{\partial D^A}, \tag{8}$$

with generic coefficients $H_{BC}^A$ and $K_{BC}^A$ to be assigned particular values later. Vertical gradients of basis vectors are likewise found in terms of generic coefficients $V_{BC}^A$ and $Y_{BC}^A$:

$$\nabla_{/\partial D^\mu} \frac{\partial}{\partial D^A} = V_{BC}^A \frac{\partial}{\partial D^A}, \quad \nabla_{/\partial D^\mu} \frac{\delta}{\delta X^A} = Y_{BC}^A \frac{\delta}{\delta X^A}. \tag{9}$$

The above descriptions pertain to both pseudo-Finsler space and Finsler space. The latter type of space is a subset of the former. A Finsler space is further characterized by existence of a $C^\infty$ fundamental scalar function $\mathcal{L}(X, D)$ at every point of $\Omega \setminus \{0\}$, homogeneous of degree one in $D$. The metric tensor, spray connection coefficients, and Cartan tensor are all obtained from differentiation of this function:

$$G_{AB} = \frac{1}{2} \partial_A \partial_B (\mathcal{L}^2), \quad G_B^A = \gamma_{BC}^A D^C - C_{BC}^A \gamma_{DE}^C D^D D^E = \Gamma_{BC}^A D^C, \tag{10}$$

$$C_{ABC} = \frac{1}{4} \partial_A \partial_B \partial_C (\mathcal{L}^2).$$

Two specific connections often encountered in the literature are, e.g. [13]:

- Chern–Rund connection: Eq. (11) holds and $N_B^A = G_B^A, H_{BC}^A = K_{BC}^A = \Gamma_{BC}^A, V_{BC}^A = Y_{BC}^A = 0$;
- Cartan connection: Eq. (10) holds and $N_B^A = G_B^A, H_{BC}^A = K_{BC}^A = \Gamma_{BC}^A, V_{BC}^A = Y_{BC}^A = C_{BC}^A$.

Let $(\cdot)_C$ denote horizontal covariant differentiation in a coordinate chart $\{X^C\}$. Then when either of these two connections is used, the horizontal covariant derivative of the metric vanishes identically: $G_{ABC} = 0$. Finsler space reduces to Riemannian space when $G_{AB}(X, D) \rightarrow G_{AB}(X)$. Finsler space reduces to locally Minkowskian space when $\mathcal{L}(X, D) \rightarrow \mathcal{L}(D)$ [13,56].

In pseudo-Finsler geometry, $\mathcal{L}$ need not exist over the (whole) space. Even in pseudo-Finsler spaces, the metric $G$ is usually taken to be homogeneous of degree zero with respect to $D$. Such an assertion yields dependence of the metric on the direction of $D$ but not its magnitude. In this work, as discussed in [22], the homogeneity assumption is too restrictive for many applications. The homogeneity requirement is relaxed here, as in [19,22,23] to permit $D$ to differ from unit length and allow a metric to depend potentially on the magnitude and direction of $D$. The geometric space is therefore referred to as one of generalized (pseudo)-Finsler geometry, and the director vector is referred to as a state vector or internal state vector. Regardless of existence of $\mathcal{L}$ or homogeneity of $G$ with respect to $D$,
Cartan’s tensor components and coefficients of Levi-Civita, Chern–Rund, and Cartan connections are still defined via Eqs. (3), (4), and (6); these are referred to by the same names regardless of whether or not the space is Finsler, pseudo-Finsler, or more general. Various modifications of classical or strict Finsler geometry not requiring homogeneity of the metric were earlier analyzed in [54], along with generalized Finsler connections defined independently of a fundamental scalar or even a metric. Many identities in Finsler geometry [55, 56] like (10) require homogeneity, and sometimes more strongly, existence of \( \mathcal{L} \). However, derivations invoked later in this paper do not require that either of such conditions holds, as discussed in [22].

Denote by \( d\mathbf{X} \) a differential line element on \( \mathfrak{M} \) and \( d\mathbf{D} \) a corresponding element on \( \Omega \). Squared differential line lengths with respect to the Sasaki metric in (2) are

\[
|d\mathbf{X}|^2 = \langle d\mathbf{X}, G d\mathbf{X} \rangle = G_{AB} dX^A dX^B, \\
|d\mathbf{D}|^2 = \langle d\mathbf{D}, G d\mathbf{D} \rangle = G_{AB} dA^A dB^B. \tag{11}
\]

Scalar volume elements and volume forms of \( \mathfrak{M} \) are defined as [57]

\[
dV = \sqrt{G} dX^1 dX^2 dX^3, \quad d\Omega = \sqrt{G} dX^1 \wedge dX^2 \wedge dX^3; \tag{12}
\]

the differential area form corresponding to a compact region of \( \mathfrak{M} \) is

\[
\Omega = \sqrt{\beta} dU^1 \wedge dU^2; \quad \left[ X^A = X^A(U^\alpha)(\alpha = 1, 2); \right. \beta^A = \frac{\partial X^A}{\partial U^\alpha}, \beta = \det(\beta^A G_{AB} \beta^B) \bigg]. \tag{13}
\]

The following identities hold:

\[
\delta_A (\ln \sqrt{G}) = \Gamma_B^A_{AB}, \quad (\sqrt{G})|_A = \partial_A (\sqrt{G}) - N^B_A \partial_B (\sqrt{G}) - \sqrt{G} H_{AB} B^B. \tag{14}
\]

Let \( V^A(X, D) \Omega(X, D) \) be a 2-form, and let \( V^A \) be contravariant components of vector field \( \mathbf{V} = V^A \frac{\partial}{\partial X^A} \). Choose the horizontal connection to be one for which

\[
H_{AB}^B = \Gamma_A^B_{AB} \to (\sqrt{G})|_A = 0. \tag{15}
\]

Then in a coordinate chart \( \{X^A\} \), the version of Stokes’ theorem used herein is due to Rund [57]:

\[
\int_{\mathfrak{M}} [V^A|_A + (V^A C^C_{BC} + \delta_B V^A) D^B_A]|d\Omega = \oint_{\partial \mathfrak{M}} V^A N_A \Omega. \tag{15}
\]

Regarding notation, \( N_A \) is the unit outward normal to \( \partial \mathfrak{M} \). \( V^A|_A = \delta_A V^A + V^A H_{BA}^B \) is the horizontal divergence of \( \mathbf{V} \), and \( D^B_A = \partial_A D^B + N^B_A \). It can be verified that (15) holds in generalized pseudo-Finsler space so long as the horizontal gradient of the metric vanishes [22]. Using notation introduced in [26], a special covariant derivative operation \( (\cdot)|_A \) is defined in a reference coordinate chart as

\[
(\cdot)|_A = (\cdot)|_A + [(\cdot) C^C_{BC} + \delta_B (\cdot)]D^B_A \Rightarrow \int_{\mathfrak{M}} (\cdot)|_A d\Omega = \oint_{\partial \mathfrak{M}} (\cdot) N_A \Omega. \tag{16}
\]

Thus (16) conveniently enables presentation of Stokes’ theorem in a compact form.


2.1.2. Deformed configuration geometry

The current configuration or spatial configuration corresponds to an instance in time at which the solid body is mechanically deformed. A pseudo-Finsler geometric framework is constructed in exact parallel to that of Sec. 2.1.1, with the caveat that notation differs according to conventions of continuum mechanics. Specifically, deformed coordinates and their indices are here written in lower-case rather than capitals.

A differential manifold \( \mathfrak{m} \) of spatial dimension 3 is identified with the body embedded in ambient Euclidean 3-space. Let \( x \in \mathfrak{m} \) denote a spatial point, and let \( \{ x^a \} (a = 1, 2, 3) \) denote a chart on \( \mathfrak{m} \). Attached to each point is the internal state vector \( \mathbf{d} \), with secondary coordinates \( \{ d^a \} (a = 1, 2, 3) \), which can be associated with manifold \( \mathfrak{u} \) of dimension 3. The state vector \( \mathbf{d} \) need not be of unit length. As in \([54]\), define \( \mathfrak{z} = (\mathfrak{j}, \pi, \mathfrak{m}, \mathfrak{u}) \) as a fiber bundle of total (pseudo-Finsler) space \( \mathfrak{j} \) (dimension 6), with \( \pi : \mathfrak{z} \rightarrow \mathfrak{m} \) the projection and \( \mathfrak{u} \) is the fiber. A composite chart on \( \mathfrak{z} \) is \( \{ x, d \} \). The natural or holonomic basis on \( \mathfrak{z} \) is \( \{ \frac{\partial}{\partial x^a}, \frac{\partial}{\partial d^a} \} \). With \( n_b^a(x, d) \) spatial nonlinear connection coefficients, non-holonomic basis vectors are

\[
\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - n_b^a \frac{\partial}{\partial d^b}, \quad \delta d^a = dd^a + n_b^a dx^b. \tag{17}
\]

The set \( \{ \frac{\partial}{\partial x^a}, \frac{\partial}{\partial d^a} \} \) serves as a local basis for \( T_z \); \( \{ dx^a, \delta d^a \} \) for \( T^*_z \). The spatial Sasaki metric is the symmetric tensor

\[
g(x, d) = g_{ab}(x, d) dx^a \otimes dx^b + g_{ab}(x, d) dd^a \otimes dd^b \tag{18}
\]

with determinant \( g(x, d) = \det[g_{ab}(x, d)] \). Spatial differentiation in coordinates follows the compact notation

\[
\partial_a(\cdot) = \frac{\partial(\cdot)}{\partial x^a}, \quad \delta_a(\cdot) = \frac{\partial(\cdot)}{\partial d^a}, \quad \delta_a(\cdot) = \frac{\delta(\cdot)}{\delta x^a} = \partial_a(\cdot) - n_b^a \delta_b(\cdot). \tag{19}
\]

The Levi-Civita connection coefficients on \( \mathfrak{m} \) are

\[
\gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_c g_{bd} + \partial_b g_{cd} - \partial_d g_{bc}) = g^{ad} \gamma_{bcd}. \tag{20}
\]

Cartan’s tensor is defined as

\[
c^a_{bc} = \frac{1}{2} g^{ad} (\partial_c g_{bd} + \partial_b g_{cd} - \partial_d g_{bc}) = g^{ad} c_{bcd}. \tag{21}
\]

Equivalent horizontal coefficients of the Chern–Rund and Cartan connections are

\[
\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_c g_{bd} + \partial_b g_{cd} - \partial_d g_{bc}) = g^{ad} \Gamma_{bcd}. \tag{22}
\]

The spray and canonical nonlinear connection coefficients (when \( n_b^a = g_b^a \)) are

\[
g^a = \frac{1}{2} \gamma^a_{bc} dx^b \otimes dx^c, \quad g^a = \delta_b g^a. \tag{23}
\]

Horizontal gradients of basis vectors are, in terms of generic connection coefficients \( H_{bc}^a \) and \( K_{bc}^a \),

\[
\frac{\nabla}{\delta x^a} \frac{\delta}{\delta x^c} = H_{bc}^a \frac{\delta}{\delta x^c}, \quad \frac{\nabla}{\delta x^a} \frac{\partial}{\partial d^c} = K_{bc}^a \frac{\partial}{\partial d^c}. \tag{24}
\]
Similarly, in terms of general vertical connection coefficients $V_{bc}^a$ and $Y_{bc}^a$, vertical gradients of basis vectors are

$$\nabla_{\partial/\partial d^b} \frac{\partial}{\partial d^c} = V_{bc}^a \frac{\partial}{\partial x^a}, \quad \nabla_{\partial/\partial d^b} \frac{\delta}{\delta x^c} = Y_{bc}^a \frac{\delta}{\delta x^a}. \quad (25)$$

Classifications regarding strict Finsler versus pseudo-Finsler space of Sec. 2.1.1 and their generalizations hold for the spatial manifold, as do definitions for spatial Chern–Rund and Cartan connection coefficients. Line and volume elements exist analogously to those in Sec. 2.1.1. An area form akin to (13) can be introduced, and spatial versions of Stokes’ theorem and Rund’s divergence theorem in (15) hold.

2.1.3. Deformation kinematics

The motion and its inverse from $\mathfrak{m}$ to $\mathfrak{m}$ and vice versa are the $C^2$ functions

$$x^a(X, D) = \varphi^a[X, D(X)], \quad X^A(x, d) = \Phi^A[x, d(x)]. \quad (26)$$

Time does not enter as an explicit independent variable yet to maintain a compact presentation of the theory, but it will be introduced later in Sec. 2.2.3. Incorporation of the internal state ($D$ or $d$) in these motion functions in part delineates Finsler kinematics [19, 54, 58, 59] from classical finite kinematics in Riemannian geometry.

State vector mappings are

$$d^a(X, D) = \theta^a[X, D(X)], \quad D^A(x, d) = \Theta^A[x, d(x)]. \quad (27)$$

These mappings may be non-affine transformations, in general [23], differing from those in [54], for example.

The deformation gradient is a tangent map defined as the partial derivative of motion, referred to the non-holonomic basis [19]:

$$\mathbf{F}(X, D) = F^a_A(X, D) \frac{\delta}{\delta x^a} \otimes dX^A = \frac{\partial \varphi^a(X, D)}{\partial X^A} \frac{\delta}{\delta x^a} \otimes dX^A = \frac{\partial x(X, D)}{\partial X}. \quad (28)$$

The inverse mapping from spatial to referential tangent spaces analogously obeys

$$\mathbf{f}(x, d) = f^A_a(x, d) \frac{\delta}{\delta X^A} \otimes dx^a = \frac{\partial \Phi^A(x, d)}{\partial x^a} \frac{\delta}{\delta X^A} \otimes dx^a = \frac{\partial X(x, d)}{\partial x}. \quad (29)$$

Deformation gradients $\mathbf{F}$ and $\mathbf{f}$ are invertible with positive determinants, and are mutual inverses of one another at coincident points on $\mathfrak{m}$ or $\mathfrak{m}$, i.e. $F^a_A f^A_a = \delta^a_b$ and $f^A_a F^a_A = \delta^A_B$. The director deformation gradients are defined as the following partial derivatives of (27) with components referred to the non-holonomic basis [23]:

$$\vartheta^a(X, D) = \theta^a_A(X, D) \frac{\partial}{\partial d^a} \otimes \delta D^A = \frac{\partial \varphi^a(X, D)}{\partial D^A} \frac{\partial}{\partial d^a} \otimes \delta D^A = \frac{\partial x(X, D)}{\partial D}. \quad (30)$$

$$\vartheta^a = \partial_A \theta^a;$$
Importantly, inelastic components depend functionally only on the internal state, and the entire inelastic two-point tensor $F$ is allowed further dependence on $X$.
Generalized pseudo-Finsler geometry

only through possible dependence of its basis vectors on $X$. Greek indices describe a generally anholonomic space [24, 25] often called the intermediate configuration in plasticity literature. Local integrability conditions for the total deformation gradient, an application of Poincaré’s lemma, are

$$ \partial_B \partial_A \varphi^a = \partial_B \partial_A \varphi^a \Leftrightarrow (F^E)^a_{\alpha} \partial_A (F^D)^{\alpha}_{B} - \partial_B (F^D)^a_{\alpha} = (F^D)^{\alpha}_{B} \partial_B (F^E)^a_{\alpha}. \quad (38) $$

Neither $F^E$ nor $F^D$ nor their inverses need be integrable individually to any vector-valued fields. The non-existence of such fields can be physically linked to density tensors of structural defects in crystalline materials, e.g. as derived in [3–5, 7, 9]. A complementary multiplicative split of the director gradient function $\dot{\vartheta}$ of (30) was introduced in [23] but is not needed here.

Multiplicative decompositions of the referential Sasaki metric tensor of (2) and its inverse into position- and structure-dependent parts are invoked later:

$$ G_{AB}(X,D) = \bar{G}_{AC}(X) \hat{G}^C_B(D), \quad G^{AB}(X,D) = \bar{G}^{AC}(X) (\hat{G}^{-1})^B_C(D). \quad (39) $$

Analogous functional decompositions into position- and microstructure-dependent parts are used for the spatial Sasaki metric of (18) and its inverse components:

$$ g_{ab}(x,d) = \bar{g}_{ac}(x) \hat{g}^c_b(d), \quad g^{ab}(x,d) = \bar{g}^{ac}(x) (\hat{g}^{-1})^b_c(d). \quad (40) $$

A metric tensor is also necessarily introduced for the geometry of the intermediate configuration [25]. This symmetric tensor is likewise conveniently split multiplicatively into terms depending on position and structure [24]:

$$ g_{\alpha\beta}(X,D) = \bar{g}_{\alpha\gamma}(X) \hat{g}^\gamma_\beta(D), \quad g^{\alpha\beta}(X,D) = \bar{g}^{\alpha\gamma}(X) (\hat{g}^{-1})^\gamma_\beta(D); \quad \tilde{g} = \det(g_{\alpha\beta}) = 1/\det(g^{\alpha\beta}). \quad (41) $$

For the total intermediate metric and intermediate structure-independent metric at a material point $X \in \mathfrak{M}$, the following choice [25] will be implemented later in Sec. 4:

$$ g_{\alpha\beta}(X,D) = \delta^A_\alpha \bar{G}_{AB}(X,D) \delta^B_\beta = \delta^A_\alpha \bar{G}_{AB}(X) \tilde{g}_\beta^A(D) \delta^B_\beta, \quad \tilde{g}_{\alpha\beta}(X) = \delta^A_\alpha \bar{G}_{AB}(X) \delta^B_\beta. \quad (42) $$

The convenience of this choice of identical reference and intermediate metric tensor components, one of several advocated in [24, 25, 62] in the context of Riemannian geometry, was recognized in [23]. This choice ensures invariance of components of the elastic deformation measure $C^E$ in [50] under transformation spatial coordinates and, for cases in which inelastic deformation does not occur, reduces the description to one similar to nonlinear elasticity wherein curvilinear intermediate and referential coordinates become coincident.
Local volume element $d\tilde{v}$ and volume form $d\tilde{\omega}$ on the intermediate space are obtained by direct analogy with those in (33):

$$d\tilde{v} = \left\{ \det \left[ (F^D)_{A}^{a} \right] \sqrt{\tilde{g}/G} \right\} dV = J^D dV,$$

$$d\tilde{v} = \left\{ \det \left[ (F^{E^{-1}})_{A}^{a} \right] \sqrt{\tilde{g}/g} \right\} dv = J^E dv;$$

$$d\tilde{\omega} = J^D d\Omega = J^E d\omega. \quad (43)$$

Jacobian determinants of inelastic and elastic deformation tensors are

$$J^D = \frac{1}{J_D} = \left\{ \det \left[ (F^D)_{A}^{a} \right] \sqrt{\tilde{g}/G} \right\} = \frac{d\tilde{v}}{dV},$$

$$J^E = \frac{1}{J_E} = \left\{ \det \left[ (F^{E^{-1}})_{A}^{a} \right] \sqrt{\tilde{g}/g} \right\} = \frac{dv}{dV}. \quad (44)$$

For the advocated choice $\tilde{g} = G$. Derivations pertaining to the specific case of purely volumetric inelastic deformation, i.e. a perfectly spherical form of $F^D$, are reported in further detail in [23], but are insufficient to describe the more complex case of torsional deformation in later sections. A multiplicative decomposition of the total deformation gradient into terms accounting for elasticity, dislocation plasticity, and ductile damage due to void mechanisms was introduced in [63], where the latter term is isotropic for nucleation and growth of a uniform distribution of spherical voids within a material element.

The decomposition (36) is most often motivated from plasticity theory, whereby evolution of the inelastic term is irreversible, i.e. fully dissipative. A more general interpretation allows the second term $F^E$ to account for deformation mechanisms that may be partially or even fully reversible, for example bowing (but not breaking free) of dislocation segments, certain twinning behaviors [64], and some phase transformations such as observed in shape memory alloys [65]. The distinction between $F^E$ and $F^D$ is that the former is associated with homogeneous microscopic deformation of the lattice induced by mechanical stress, while the latter is due to changes in the microstructure from defects or structural transformations.

2.2. Thermodynamics and balance laws

The variational framework for quasi-static equilibrium in pseudo-Finsler geometry developed in [19, 22, 23] is reviewed in brief in Sec. 2.2.1 followed by ramifications of the multiplicative description in Sec. 2.2.2. Explicit time dependence of fields is introduced in Sec. 2.2.3 with particular emphasis on the kinetic relation for the internal state.

2.2.1. Variational principle

The action integral for a compact region of $\mathcal{M}$ with boundary $\partial \mathcal{M}$ is denoted by $\Psi$. Let $p = p_a dx^a$ be a mechanical load vector, specifically a force per unit reference area. Let $z = z_A \delta D^A$ designate a thermodynamic force conjugate to the internal
state vector. The variational principle used here, as in prior works [19, 22, 23], is

$$\delta \Psi(\varphi, D) = \oint_{\partial \mathcal{M}} \left( \langle p, \delta x \rangle + \langle z, \delta D \rangle \right) \Omega. \quad (45)$$

The scalar product operation is $\langle \cdot, \cdot \rangle$. Denote by $\psi$ the free energy density potential. Then (45) is further expressed as

$$\delta \int_{\mathcal{M}} \psi d \Omega = \oint_{\partial \mathcal{M}} \left[ p_a \delta x^a + z_A \delta (D^A) \right] \Omega. \quad (46)$$

Regarding notation, the first variation of components of internal state vector $D$ is enclosed in parentheses to avoid confusion with basis vector $\delta D^A$. Energy density per unit reference volume on $\mathcal{M}$ is of the functional form

$$\psi = \psi(F, D, \nabla D, G) = \psi(F^a_A, D^A, D^A_B, G_{AB}). \quad (47)$$

As remarked in [19], generalized continuum theories of materials with microstructure [66] such as phase field theories [67, 50] produce the physical motivation for the form of (47). Internal state vector $D$ is viewed as a vector-valued set of order parameter(s) that may transform between reference and spatial configurations in a special manner to be discussed later by application in Sec. 3. The transformation law for $D$ between different (partially) coincident coordinate charts covering the same referential configuration space $(\mathcal{M}, \Omega)$ is that of generalized Finsler geometry [19, 54].

In mechanics, internal variables are traditionally introduced in the context of ordinary (differential) evolution equations. The introduction of internal variables with diffusive transport, i.e. gradients of internal variables as arguments of the free energy potential, was pioneered in [68, 69].

Thermodynamic forces are acquired from the first variation of (47):

$$\delta \psi = \frac{\partial \psi}{\partial F_A^a} \delta F_A^a + \frac{\partial \psi}{\partial D^A} \delta (D^A) + \frac{\partial \psi}{\partial D^A_B} \delta D_A^B + \frac{\partial \psi}{\partial G_{AB}} \delta G_{AB}$$

$$= P_A^a \delta F_A^a + Q_A \delta (D^A) + Z^B_A \delta D^A_B + S^{AB} \delta G_{AB}. \quad (48)$$

Spatial coordinate invariance requires that dependence on $F$ is only through $C$ of (51), for example:

$$\psi = \psi[C(F, g), D, \nabla D, G] = \psi(C_{AB}, D^A, D^A_B, G_{AB}). \quad (49)$$

The first Piola–Kirchhoff stress $P_A^a$ (oriented force per unit reference area) and Cauchy stress $\sigma^{ab}$ (oriented force per unit current area) obey a local angular momentum balance leading to traditional symmetry of the latter:

$$P_A^a = 2g_{ab} F_B^b \frac{\partial \psi}{\partial C_{AB}}.$$

$$\sigma^{ab} = j g^{ac} P_c^a F^b_A = 2j F_A^a F_B^b \frac{\partial \psi}{\partial C_{AB}} = \sigma^{ba}. \quad (50)$$
The following identities are obtained with variation \( \delta(\cdot) \) executed at fixed \( X \) and variable \( D \) \[19, 23\]:

\[
\delta F_A^c = \delta_A(\phi^a) + \tilde{\partial}_B \tilde{\partial}_C \phi^a N^B_A \delta(D^C),
\]

\[
\delta D^A_B = [\delta(D^A)]_B - (\tilde{\partial}_C N^A_B - \tilde{\partial}_C K^A_{BD} D^D) \delta(D^C);
\]

\[
\delta(d\Omega) = G^{AB} \tilde{\partial}_C \tilde{\partial}_A \delta(D^C) d\Omega.
\]

Euler–Lagrange equations are derived as follows \[19\]. First, (48), (51), and (52) are substituted into (46). The divergence theorem \[15\] and repeated integration by parts are next used to produce an integral/global form of \[16\]. Localizing this result and then requiring the equality to hold for admissible variations \( \delta \phi \) and \( \delta D \), the Euler–Lagrange equations in \( \mathfrak{M} \) and natural boundary conditions on \( \partial \mathfrak{M} \) are finally obtained:

\[
\partial_A P^A_a + P^B_a H^A_{AB} - P^c_e H^c_{ba} (F^b_a - N^B_A \tilde{\partial}_D \phi^b) + P^A_a N^B_A C^C_{BC} + (P^c_a C^c_{BC} + \tilde{\partial}_B P^a_A) \partial_A D^B = 0,
\]

\[
\partial_A Z^A_a + Z^B_a H^A_{AB} - Z^c_A H^c_{AC} + \tilde{\partial}_B Z^a_A \partial_A D^B + Z^B_A (\tilde{\partial}_C N^A_B - \tilde{\partial}_C K^A_{BD} D^D + \delta^A_C C^D_{EB} E^B) + P^A_a \tilde{\partial}_B \tilde{\partial}_C \phi^a \partial_A D^B - (S^{AB} + \psi^{GAB}) \tilde{\partial}_C G_{AB} = Q_C;
\]

\[
p_a = P^A_a N^A_a, \quad z_A = Z^B_A N^B_a.
\]

Equation (53) is the local balance of linear momentum, i.e. macroscopic momentum or mechanical stress equilibrium, in the absence of body forces and inertial forces. Equation (54) is the local balance of director momentum or micro-momentum for static equilibrium of the internal state. The operation (55) has been used in the above presentation of (53) for covariant differentiation of the two-point stress tensor \( \mathbf{P} = P^a_a dx^a \otimes \Box^a \). Reductions of balance equations for (pseudo)-Riemannian, (pseudo)-Minkowskian, and Cartesian spaces are derived in \[19\].

2.2.2. Multiplicative thermodynamics

The total free energy density \( \psi \) of (17) is decomposed additively into an elastic strain energy density potential \( W \) and a microstructure- or internal state-dependent energy density potential \( f \), both measured per unit initial volume. The following functional forms are used, similarly to those invoked in prior Finsler-geometric models \[19, 21, 23\] and phase field theory \[50, 64, 70, 71\]:

\[
\psi(F_A^c, D^A, D^A_{|B}, G_{AB}) = W[(F^E)_a, D^A, \delta_{ab}] + f(D^A, D^A_{|B}, G_{AB}).
\]

The elastic deformation can be written as follows using functional conditions (55) and (57):

\[
(F^E)_a[F, F^D(D)] = F_A^a [X, D(X)] (F^{D^{-1}})^A_a [D(X)].
\]
Using (56) and (57) in conjunction with assumption (42) for the intermediate metric components \(\bar{g}_{\alpha\beta}\), the conjugate thermodynamic forces introduced in (48), at a fixed material point \(X \in \mathfrak{M}\), are

\[
P^A_a = \frac{\partial \psi}{\partial F^a_A} = \frac{\partial W}{\partial F^a_A} \quad Q_A = \frac{\partial \psi}{\partial D^A} = \frac{\partial W}{\partial D^A} + \frac{\partial f}{\partial D^A} - P^B_a (F^E)^a_{\alpha} \frac{\partial (F^D)^\alpha}{\partial D^B},
\]

(58)

\[
Z^B_A = \frac{\partial \psi}{\partial D^A]|_B = \frac{\partial f}{\partial D^A}|_B.
\]

Since \(\partial \psi/\partial G_{AB} \to \partial \psi/\partial \bar{G}_{AB}\) and \(\delta \bar{G}_{AB}(X) = 0\), the simplifications \(S^{AB} \to 0\) in (48) and (54) emerge. Spatial invariance of the strain energy density analogous to (49) can be ensured via letting \(W\) depend on a symmetric elastic deformation tensor \(C^E\) rather than \(F^E\), where

\[
(C^E)^{\alpha\beta} = (F^E)^a_{\alpha} \bar{g}_{ab} (F^E)^b_{\beta}, \quad \bar{C}^{AB} = F^a_A \bar{g}_{ab} F^b_B.
\]

(59)

2.2.3. Time dependence

In this work, quasi-static conditions for mechanical momentum are assumed, but dynamic evolution of the state vector field (i.e. order parameter kinetics) is enabled. These simplifications are often encountered in phase field applications \([72, 73]\). Isothermal conditions are also invoked consistently with developments in Secs. 2.2.1 and 2.2.2 so that the free energy is temperature-independent. A complete set of governing equations for the fully dynamic case — including material inertia, temperature, and entropy changes — is derived in \([26]\).

With time \(t\) acting as another independent parameter, fields introduced in Sec. 2.2.1 and earlier parts of Sec. 2.2 are generalized to include time in their mathematical arguments. For example, enhancing (26), the motion function from \(\mathfrak{M}\) to \(\mathfrak{m}\) and its inverse at time \(t\) are now, respectively,

\[
x^a(X, D, t) = \varphi^a(X, D, t), \quad X^A(x, d, t) = \Phi^A(x, d, t).
\]

(60)

Similarly, mappings of state vectors between configurations for the static case in (27) now become the more general dynamic functions

\[
d^a(X, D, t) = \theta^a[X, D(X, t), t], \quad D^A(x, d, t) = \Theta^A[x, d(x, t), t].
\]

(61)

Let \(D(\cdot)/Dt\) denote the material time derivative, defined as the partial time derivative of a quantity at a fixed material point \(X\) and at fixed internal state \(D\). The material velocity vector \(\mathbf{v}\) is defined as the material time derivative of position:

\[
\mathbf{v}(X, t) = \frac{\partial \mathbf{x}(X,t)}{\partial t} = \frac{\partial \varphi^a(X, t)}{\partial t} \frac{\partial}{\partial x^a}, \quad v^a = \frac{D x^a}{Dt}.
\]

(62)

A superposed dot is also used as notation for material time differentiation when no ambiguity arises. A special case of the material time derivative applies to the
internal state, specifically a partial derivative at constant $X$ [26]:

$$\dot{D}(X, t) = \frac{\partial D^A(X, t)}{\partial t} \frac{\partial}{\partial D^A}. \quad (63)$$

In [54], as in prior work [19], a local equilibrium equation was derived for micro-momentum, written primarily in terms of micro-forces $Q$ and $Z$. By logical extension, for irreversible kinetics, the evolution equation for internal state vector components $D^A$ is deduced by setting the residual of that equilibrium equation proportional to the negative rate of internal state as first declared in [26]:

$$\dot{D} = -L^{KC} [Q_C - \partial_A Z^A_C - Z^B_C H^A_B + Z^A_B H^B_C - \partial_B Z^C_A \partial_A D^B - Z^B_A (\partial_C N^A_B - \partial_C K^A_B D^B + \delta^A_C D^E D^E_B) - P^A_a \partial_B \partial_C \varphi^a \partial_A D^B + (S^{AB} + \psi^{G^{AB}}) \partial_C G^{AB}], \quad (64)$$

Here, $L^{KC}$ is a positive-definite matrix of material constants dictating the time scale for microstructure kinetics. Equation (64) suggests that order parameter(s) evolve in time so that at equilibrium, the term in square braces vanishes in accordance with the static director momentum equation (54). More transparently, (64) can be expressed in abbreviated form:

$$\dot{D} = -L^{KC} \left[ \frac{\partial \varphi}{\partial D^C} - \nabla_A \left( \frac{\partial \varphi}{\partial (\partial_A D^C)} \right) + \cdots \right], \quad (65)$$

where various higher-order and nonlinear terms are truncated for presentation purposes. This demonstrates that (64) is the generalized pseudo-Finsler analog to the time-dependent Ginzburg–Landau or Allen–Cahn equations [73, 74].

2.2.4. Summary

The theoretical model becomes complete for solving particular boundary value problems following assignment of several features. A metric tensor $G$ is introduced, from which all connection coefficients are derived via differentiation using relations listed in Sec. 2.1.1. Horizontal and vertical connection coefficients in (8) and (9) must be chosen, e.g. those of the Chern–Rund connection or Cartan’s connection as used in prior work [19, 22]. The current configuration is equipped with a metric tensor $g$ from which connection coefficients are similarly derived via equations in Sec. 2.1.2. The free energy function $\psi$ in (47) for the particular class of material under consideration must be assigned a more specific form. Constitutive equations for inelastic components of the deformation gradient, $F^D$, should be imposed by the modeler that relate inelastic kinematics to the internal state.

With boundary conditions assigned on $\partial M$, (53) and (64) are six coupled nonlinear partial differential equations for six unknown time-dependent field components $\varphi^a[X, D(X, t), t]$ and $D^A(X, t)$ where $a, A = 1, 2, 3$ in general. When multiplicative kinematics and thermodynamics are invoked, functions $F^D, W$, and $f$ of Sec. 2.2.2 apply. Substitution of thermodynamic forces of (58) into (53) and (64) and using $F^E = (\partial \varphi / \partial X)(F^D)^{-1}$ then gives more particular versions of the same six partial
differential equations in six unknowns \( \varphi^a(X, D, t) \) and \( D^A(X, t) \). These equations are next specifically derived for generalized torsional deformation protocols in Sec. 3.

3. Differential-Geometric Theory: Generalized Torsion

The general continuum theory — with multiplicative kinematics — of Sec. 2 is now specialized to modeling torsional deformation of a cylindrical sample. Simultaneously, the sample may be extended or compressed along its axis of transverse isotropy, and the lateral surfaces of the cylinder may undergo radial deformation or be constrained not to do so. The inelastic deformation consists of possible volumetric and shearing terms, and the internal state accounts for possible softening of the tangent elastic modulus that accompanies generation and motion of defects, for example. The objective of Sec. 3 is introduction of the requisite assumptions on geometry, kinematics, and constitutive behavior and commensurate reduction of the general governing equations of Sec. 2 to particular partial differential equations that will be solved numerically in Sec. 4. The torsion problem studied here has not been analyzed in any other prior work, e.g. [19, 21–23, 26], incorporating similar theory.

3.1. Geometry and deformation

The present class of problems considers a cylindrical material body of initial external radius \( R_0 \) and length \( L_0 \). The material manifold is then \( \mathcal{M} : R \in [0, R_0], \Theta \in (-\pi, \pi], Z \in [0, L_0] \), with \( \Theta \) the angular coordinate and \( Z \) the axial coordinate. The internal structure field is physically associated with twisting or torsion in the positive \( \Theta \)-direction. To maintain axisymmetric conditions, this field varies potentially with \( R \) and \( Z \) but not \( \Theta \), and also evolves with time \( t \). Thus, components of the internal state vector are of the form

\[
\{D^1, D^2, D^3\} = (0, D^2, 0), \quad D = D^2 = D^\Theta = D(R, Z, t) \geq 0. \quad (66)
\]

The referential coordinate chart covering \( \mathcal{M} \) is

\[
\{X^1, X^2, X^3\} = \{R, \Theta, Z\}. \quad (67)
\]

Define \( \check{G} = \check{G}(X) \) to be the usual metric tensor of Euclidean space in cylindrical coordinates \( [23, 24] \). The multiplicatively separable form of the total metric tensor \( G(X, D) \) in (69) is prescribed as

\[
G(X, D) = \check{G}(X)\hat{G}(D) = \begin{bmatrix}
\check{G}_{11} & 0 & 0 \\
0 & \check{G}_{22}(X) & 0 \\
0 & 0 & \check{G}_{33}
\end{bmatrix}
\begin{bmatrix}
\hat{G}^1_1(D) & 0 & 0 \\
0 & \hat{G}^2_2(D) & 0 \\
0 & 0 & \hat{G}^3_3(D)
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
0 & R^2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
B(D) & 0 & 0 \\
0 & B(D) & 0 \\
0 & 0 & B(D)
\end{bmatrix}. \quad (68)
\]
The cylindrical metric is scaled isotropically by the scalar factor \( B \), a conformal transformation or Weyl-type scaling [75], as justified in prior work [19, 23], specifically as follows:

\[
B(D) = \exp\left(\frac{kD^2}{3l^2}\right) = \exp\left(\frac{k\eta^2}{3}\right)
\]

\[\Rightarrow \hat{G}^A_B(D) = B(D)\delta^A_B = B[\eta(D)]\delta^A_B = \exp\left(\frac{k\eta^2}{3}\right)\delta^A_B.\]  \hspace{1cm} (69)

Scalar \( B > 0 \) is an exponential function of the internal state. Also introduced are the regularization length \( l \) and the scaling factor \( k \), both of which are chosen by the modeler for the particular material under consideration. By construction, \( l > 0 \), while \( k \) may be positive or negative depending on whether the material expands or contracts with evolution of the internal state. The dimensionless order parameter is

\[\eta = D/l, \quad 0 \leq \eta \leq 1.\]  \hspace{1cm} (70)

Physically, \( D \) measures microscopic displacements of internal structure in the direction of torsional deformation, for example emerging from glide of dislocations, twisting disclinations, or microscopic fractures and sliding of crack surfaces. The order parameter is a dimensionless measure of \( D \). When \( D = l \), the micro-deformation is said to have achieved saturation in the sense that total loss of torsional rigidity has occurred, as will be explained later in the context of the constitutive model. For a ductile material, this would correspond to rupture, while for a brittle solid, it would correspond to macroscopic fracture. The determinant of the total metric tensor is

\[G(X, D) = \det[\mathbf{G}(X, D)] = R^2B^3[D(R, Z, t)] = R^2\exp[k[\eta(R, Z, t)]^2].\]  \hspace{1cm} (71)

Connection coefficients are now obtained from the metric assigned in (68) and (69). Christoffel symbols of the second kind are of the general form

\[\gamma^A_{BC} = G^{AD}\gamma_{BCD} = \hat{G}^{AD}\hat{\gamma}_{BCD} = \hat{\gamma}^A_{BC}.\]  \hspace{1cm} (72)

Nonzero Christoffel symbols resulting from (68) are then calculated as follows with \((1, 2, 3) \leftrightarrow (R, \Theta, Z)\):

\[\gamma^\Theta_R = \gamma^R_\Theta = 1/R, \quad \gamma^R_\Theta = -R.\]  \hspace{1cm} (73)

With \( B' = dB/dD = (1/l)dB/d\eta = 2k\eta B/(3l) \), components of Cartan’s tensor in (5) are computed directly as follows, with others all vanishing:

\[C_{112} = C_{332} = -B'/2, \quad C_{222} = R^2B'/2,\]

\[C_{121} = C_{211} = C_{233} = C_{323} = B'/2.\]  \hspace{1cm} (74)

The trace in the circumferential direction of Cartan’s tensor is needed later in the governing equations:

\[C^A_{2A} = C^1_{21} + C^2_{22} + C^3_{23} = G^{AB}C_{AB} = 3B'/(2B) = k\eta/l.\]  \hspace{1cm} (75)
The Chern–Rund connection is used, similarly to prior work \[19,22,23\]. However, in contrast to those prior works, the spray and canonical nonlinear connection coefficients do not vanish identically for the torsion problem. Rather, the nonzero components are, from (1),

\[
G^1 = G^R = -R \left(\frac{D^2}{2} - \frac{\eta^2}{2}\right), \quad N^1 = G^R = -R \delta = -R \delta \eta. \tag{76}
\]

The delta derivative in (3) becomes

\[
\delta_1(\cdot) = \partial_1(\cdot) = \partial_R(\cdot), \quad \delta_2(\cdot) = \partial_2(\cdot) + R \delta \partial_1(\cdot) = \partial_\theta(\cdot), \quad \delta_3(\cdot) = \partial_3(\cdot) = \partial_Z(\cdot). \tag{77}
\]

Since partial coordinate differentiation and delta differentiation are equivalent [note \(\delta_1(\cdot) \to 0\) for the form in (63)], the horizontal Chern–Rund connection coefficients \(K_{BC}^A = \Gamma_{BC}^A = \gamma_{BC}^A\); \(\gamma_{BC}^A\) are all equal to the Levi-Civita coefficients derived from \(\bar{G}_{AB}\) in (73). The vertical coefficients vanish by definition of the Chern–Rund connection [18,55]:

\[
A_{BC}^A = 0. \tag{78}
\]

With the above definitions and identities at hand, the differential operation in (16) becomes, for a vector field with components \(V^A(R,Z,t) = [V^R(R,Z,t), V^\theta(R,Z,t), V^Z(R,Z,t)]\),

\[
A_{||A} = \partial_A V^A + \frac{V^R}{R} + V^A k_\eta \partial_A \eta + \bar{\partial}_\theta V^A \partial_A \theta. \tag{79}
\]

The geometric description for the deformed solid in the spatial representation is analogous to that for the reference state. The deformed body is a right circular cylinder of length \(L\) and external radius \(r_0\), and the spatial base manifold is \(\{ m : r \in [0,r_0], \theta \in (-\pi,\pi], z \in [0,L]\}\). The internal state function associated with torsional micro-deformation is circumferential and varies potentially with radial and axial coordinates and time, so \(d(x,t) = d^2(x^1, x^3, t) = d^2(r,z,t) = d(r,z,t)\). The spatial analogs of (67)–(78) are derived as follows, primarily using the notation convention of lower-case for variables in the spatial frame that are complementary to those capitalized for the reference configuration:

\[
\{ x^1, x^2, x^3 \} = \{ r, \theta, z \}, \quad \{ d^1, d^2, d^3 \} = \{ d^r, d^\theta, d^z \} = \{ 0, d, 0 \}, \quad d = d(r,z,t); \tag{80}
\]

\[
g(x,d) = \bar{g}(x) \bar{g}(d)
\]

\[
= \begin{bmatrix}
\bar{g}_{11} & 0 & 0 \\
0 & \bar{g}_{22}(x) & 0 \\
0 & 0 & \bar{g}_{33}
\end{bmatrix}
\begin{bmatrix}
\bar{g}_1^1(d) & 0 & 0 \\
0 & \bar{g}_2^2(d) & 0 \\
0 & 0 & \bar{g}_3^3(d)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 \\
0 & r^2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
b(d) & 0 & 0 \\
0 & b(d) & 0 \\
0 & 0 & b(d)
\end{bmatrix} \tag{81}
\]
\[ b(d) = \exp\left(\frac{kd^2}{3l^2}\right) = \exp\left(\frac{k\hat{\eta}^2}{3}\right) \Rightarrow \]

\[ \hat{g}_{\alpha}^\alpha(d) = b(d)\delta_{\alpha}^\alpha = b[\hat{\eta}(d)]\delta_{\alpha}^\alpha = \exp\left(\frac{k\hat{\eta}^2}{3}\right) \delta_{\alpha}^\alpha; \quad (82) \]

\[ \hat{\eta} = d/l, \quad 0 \leq \hat{\eta} \leq 1; \quad (83) \]

\[ g(x, d) = \det[g(x, d)] = r^2b^3[d(x, t)] = r^2 \exp\{k[\hat{\eta}(r, z, t)]^2\}; \quad (84) \]

\[ \begin{align*}
\gamma^a_{bc} &= g^a_{bc} = \hat{g}_{abcd} \hat{\gamma}_{bcd} = \hat{\gamma}_{bc}; \\
\gamma^a_{r\theta} &= \gamma^a_{\theta r} = 1/r, \quad \gamma^r_{\theta\theta} = -r; \\
g^1 &= g^r = -(d^2/2) = -r^2\hat{t}^2/2; \\
n^1_2 &= g^t_\theta = -r\hat{\eta}; \\
c_{112} &= c_{332} = -b'/2, \quad c_{222} = r^2b'/2, \\
c_{121} &= c_{211} = c_{233} = c_{323} = b'/2; \\
c^a_{2a} &= 3b'/(2b) = k\hat{\eta}/l; \\
H^a_{bc} &= K^a_{bc} = \Gamma^a_{bc} = \gamma^a_{bc} = \hat{\gamma}_{bc}; \\
V^a_{bc} &= Y^a_{bc} = 0.
\end{align*} \quad (85-88) \]

Notice that the same regularization length \( l \) and Weyl scaling factor \( k \) are used for both configurations since the complexity of introduction of differing parameters is not physically warranted. The order parameter field is labeled \( \hat{\eta}(x, t) \) for the spatial description versus \( \eta(X, t) \) for the reference state.

The axisymmetric deformation kinematics for combined torsion and extension or compression obey the following protocols. Circular cross-sections along the length of the cylinder, normal to the axis, remain uniform, with no necking. Axial and torsional deformations are affine functions of position along the length. Finally, as in prior work [19, 22, 23, 26], the spatial and referential order parameters or internal state components are chosen to obey the canonical transformation law for scalar fields. Mathematically, these assumptions are summarized as follows:

\[ r = r(R, t) = \hat{f}(R, t)R, \quad \theta(\Theta, Z, t) = \Theta + \gamma(t)Z, \quad z(Z, t) = \Lambda(t)Z; \quad (90) \]

\[ D = d \circ \varphi \Rightarrow \hat{\eta}[x(X, t), t] = \eta(X, t). \quad (91) \]

Consistently with (66), the order parameter field may vary with radial and axial position, but not with the angular coordinate, such that inelastic deformation
Generalized pseudo-Finsler geometry

remains axisymmetric:
\[ \eta = \eta(R, Z, t) = lD(R, Z, t). \] (92)

In (90), \( \hat{f} \) is a positive dimensionless scalar function of its arguments denoting potential radial expansion or contraction, \( \gamma \) is a measure of shear strain in dimensions of twist angle per length, and the positive dimensionless scalar \( \Lambda \) is a measure of extension (if greater than unity) or compression (if less than unity). The deformation conditions imposed above are consistent with other nonlinear elastic treatments of homogeneous torsional strain [30, 31]. More elaborate (numerical) treatments beyond the scope of the present analytical paper are necessary to address heterogeneous axial and torsional strain fields arising for localized deformations, for example. The present analysis thus considers deformation up to the point of localization but not the terminal rupture process.

The total deformation gradient of (28) is computed from (90) in natural coordinates as
\[
F(R, Z, t) = \begin{bmatrix}
\frac{\partial r(R, t)}{\partial R} & 0 & 0 \\
0 & \frac{\partial \theta(\Theta, Z, t)}{\partial \Theta} & \frac{\partial \theta(\Theta, Z, t)}{\partial Z} \\
0 & 0 & \frac{\partial z(Z, t)}{\partial Z}
\end{bmatrix}
= \begin{bmatrix}
r'(R, t) & 0 & 0 \\
0 & 1 & \gamma(t) \\
0 & 0 & \Lambda(t)
\end{bmatrix}. \] (93)

The notation \( r'(R, t) = \frac{\partial r(R, t)}{\partial R} = \frac{\partial R \hat{f}(R, t)}{\partial R} + \hat{f}(R, t) \) is used for the radial component. The following non-vanishing horizontal covariant derivatives of the internal state are also needed:
\[
\eta_R = \frac{1}{l} D_R = \partial_R \eta + K_{R\Theta}^\Theta \eta = \partial_R \eta + \frac{\eta}{R}, \\
\eta_Z = \frac{1}{l} D_Z = \partial_Z \eta. \] (94)

The total Jacobian determinant is, from (93) and noting that the ratio \( g/G = (b^3/B^3)/(r^2/R^2) = r^2/R^2 \) since \( b = B \) from (91),
\[
J = \det F \sqrt{g/G} = r' \Lambda r/R. \] (95)

Coincident reference and intermediate configuration metric tensor components are used in (42), where for cylindrical coordinates,
\[
g_{\alpha\beta}(X, D) \rightarrow g_{\alpha\beta}[R, B(D)] = \delta^A_{\alpha} \delta^B_{\beta} G_{AB}[R, B(D)]. \] (96)

Identical basis vectors are thus used with respect to components of vectors and tensors in referential and intermediate configurations.

The inelastic deformation gradient \( F^D \) consists of possible shearing deformation \( \gamma^D \) (inelastic twist per unit length) as well as dimensionless volume changes \( \alpha^D \).
Mathematically, it is of the following form:

\[
\mathbf{F}^D[D(R, Z, t)] = \begin{bmatrix}
\alpha^D[D(R, Z, t)] & 0 & 0 \\
0 & \alpha^D[D(R, Z, t)] & \alpha^D[D(R, Z, t)]
\end{bmatrix}. 
\]

The elastic deformation gradient \(\mathbf{F}^E\) is then, substituting (93) and the inverse of (97) into (57),

\[
\mathbf{F}^E = \frac{1}{\alpha^D} \begin{bmatrix}
\gamma' & 0 & 0 \\
0 & 1 & \gamma - \gamma^D \\
0 & 0 & \Lambda
\end{bmatrix}. 
\]

Inelastic and elastic volume ratios in (44) are computed as follows noting \(\tilde{g} = G\):

\[
J^D = \det \mathbf{F}^D \sqrt{\tilde{g}/G} = (\alpha^D)^3, \quad J^E = \frac{J}{J^D} = \frac{r^2 \Lambda r}{(\alpha^D)^3 R}. 
\]

The covariant elastic deformation tensor of (59) and its dimensionless trace will be needed in the constitutive model later:

\[
\mathbf{C}^E = \frac{1}{(\alpha^D)^2} \begin{bmatrix}
(r')^2 & 0 & 0 \\
0 & r^2 & r^2(\gamma - \gamma^D) \\
0 & 0 & \Lambda^2 + r^2(\gamma - \gamma^D)^2
\end{bmatrix}, 
\]

\[
\text{tr}\mathbf{C}^E = C^E_{\alpha\beta} \bar{g}^{\alpha\beta} = \frac{1}{(\alpha^D)^2} \left[ (r')^2 + \frac{r^2}{R^2} + \Lambda^2 + r^2(\gamma - \gamma^D)^2 \right].
\]

Note that \(J^E = \sqrt{\det \mathbf{C}^E / G}\).

The inelastic deformation functions \(\alpha^D(D)\) and \(\gamma^D(D)\) remain to be specified. The former is prescribed such that under purely volumetric deformation, the inelastic volume change canonically coincides with the volume change associated with Weyl scaling of the material metric tensor \([22, 23]\). This assumption in conjunction with (69) gives

\[
(\alpha^D)^3 = B^{3/2} \quad \Rightarrow \quad \alpha^D = \sqrt{B} = \exp \left( \frac{k}{6} \eta^2 \right). 
\]

Recalling (11) and (12), the length of a referential line element and the corresponding volume form are

\[
|d\mathbf{X}|^2 = \exp(k\eta^2/3)(dR \cdot dR + R^2 d\Theta \cdot d\Theta + dZ \cdot dZ), \\
\]

\[
d\Omega = \sqrt{G} dR \wedge d\Theta \wedge dZ = \exp(k\eta^2/2) R dR \wedge d\Theta \wedge dZ.
\]

Expansion occurs when \(k > 0\) and contraction occurs when \(k < 0\) since \(\eta \geq 0\). At saturation wherein \(\eta = 1\), if the volume ratio induced by microstructure changes is \(\chi\), then \(k = 2 \ln \chi\). Shearing is interpolated over the range \(\gamma^D \in [0, \gamma_0]\), with \(\gamma_0\) the material constant representing inelastic twist per unit length at rupture. A
polynomial function of $\eta$ is invoked for this interpolation analogously to that used in phase field theory [67, 72]:

$$\gamma^D(\eta) = \gamma_0\eta^2(3 - 2\eta).$$

(103)

This function demonstrates the useful property of vanishing endpoint derivatives:

$$(d\gamma^D/d\eta)|_{\eta=0} = (d\gamma^D/d\eta)|_{\eta=1} = 0.$$

3.2. Balance equations

Recall from (56) that the free energy per unit reference volume $\psi$ is additively decomposed into an elastic strain energy potential $W$ and a structure-dependent energy potential $f$. For the present class of problems, a sum of quadratic forms is invoked similarly to prior work [22, 23]:

$$\psi(F_A^a, D_A^A, D_A^A|B, G_{AB}) = W[(C^E)^a_{\beta}, D^A] + f(D_A^A, D_A^A|B, \delta_{AB}).$$

(104)

The elastic potential is one of modified compressible neo-Hookean elasticity [50, 76]:

$$W = \frac{1}{2}K(\ln J^E)^2 + \frac{1}{2}\mu \left[ \text{tr} C^E - 3 - 2\ln J^E \left( 1 + \frac{1}{3} \ln J^E \right) \right].$$

(105)

The shear modulus can degrade quadratically with increasing order parameter, while the bulk modulus $K$ can degrade in the same manner only under elastically tensile states:

$$\mu = \mu_0(1 - \zeta_0\eta)^2; \quad K = K_0(1 - \zeta_0\eta)^2 \quad \text{if } J^E \geq 1,$$

$$K = K_0 \quad \text{if } J^E < 1.$$ 

(106)

Derivatives of these effective moduli are

$$\frac{d\mu}{d\eta} = -2\zeta_0\mu_0(1 - \zeta_0\eta); \quad \frac{dK}{d\eta} = -2\zeta_0K_0(1 - \zeta_0\eta) \quad \text{if } J^E \geq 1,$$

$$K' = 0 \quad \text{if } J^E < 1.$$ 

(107)

The above functional forms, with $\mu_0$ and $K_0$ the initial shear and bulk moduli of the pristine material, are consistent with those used in phase field models of fracture [41, 174]. The bulk modulus maintains its full value for elastically compressive states so that the material does not collapse upon itself; i.e. as $\zeta_0 \eta \to 1$, the shear strength vanishes but the resistance to compression does not, as in a fluid. The parameter $\zeta_0 \in [0, 1]$ is assigned a positive value if elastic softening is active, e.g. if the material is perfectly brittle, $\zeta_0 = 1$, while if perfectly ductile with no fracture, $\zeta_0 = 0$. A value in between i.e. $0 < \zeta_0 < 1$ would correspond to some ductility and some elastic softening commensurate with fracture, void growth, or other damage mechanisms. The structure-dependent term accounting for surface energy is the following function, again similar to those used in phase field theory:

$$f = \frac{Y}{l_3}|D|^2 + \frac{Y}{l}|
abla D|^2 = \frac{Y}{l}\eta^2 + Yl[(\eta_R)^2 + (\eta_Z)^2].$$

(108)
Terms entering the second equality are computed in (94), and $\delta_{AB}$ in (104) is simply used for computation of magnitude $|\cdot|$. The energy per unit area of failed or slipped surfaces is $\Upsilon$, treated as a constant for a given material. Using the forms of $W$ and $f$ in (105) and (108), the conjugate thermodynamic forces of (58) become, at a fixed material point $X$,

\[
P_a^A = \frac{\partial \psi}{\partial F^a} = (F^{D-1})^{-1}_a^{A} \frac{\partial W}{\partial (F^{D})^a_A};
\]

\[
Q = Q_2 = Q_\Theta = \frac{\partial \psi}{\partial D} = \frac{1}{l} \left[ \frac{\partial W}{\partial \eta} - P_{a}^{A} (F^{E})^a_a \frac{\partial (F^{D})^a_A}{\partial \eta} + \frac{\partial f}{\partial \eta} \right] = \frac{1}{2} K (\ln J^E)^2 + \frac{1}{2} l^' \left[ \text{tr} C^E - 2 \ln J^E \left( 1 + \frac{1}{3} \ln J^E \right) \right] - P_{a}^{A} (F^{E})^a_a \frac{\partial (F^{D})^a_A}{\partial \eta} + 2 \frac{\Upsilon}{l} \eta; \tag{109}
\]

\[
Z_1^2 = Z_R^R = \frac{\partial \psi}{\partial D_{|R}} = \frac{1}{l} \frac{\partial f}{\partial \eta_{|R}} = 2 \Upsilon \eta_{|R}; \quad \tag{110}
\]

\[
Z_3^2 = Z_R^Z = \frac{\partial \psi}{\partial D_{|Z}} = \frac{1}{l} \frac{\partial f}{\partial \eta_{|Z}} = 2 \Upsilon \eta_{|Z}. \tag{112}
\]

Using (101) and (103), the derivative of $F^D$ with respect to internal state is

\[
\frac{\partial F^D}{\partial \eta} = \exp \left( \frac{k \eta^2}{6} \right) \begin{bmatrix}
\frac{k \eta^3}{3} & 0 & 0 \\
0 & \frac{k \eta^3}{3} & \gamma_0 [(k \eta^3/3)(3 - 2 \eta) + 6 \eta (1 - \eta)] \\
0 & 0 & k \eta^3/3
\end{bmatrix}. \tag{113}
\]

More reductions of the equations invoke the cylindrical geometry and torsional kinematics of Sec. 3.1. The first Piola–Kirchhoff stress tensor consists of five possibly nonzero components:

\[
P(R, Z, t) = \begin{bmatrix}
P_R^R(R, Z, t) & 0 & 0 \\
0 & P_{\Theta}^\Theta(R, Z, t) & P_z^Z(R, Z, t) \\
0 & P_{\Theta}^\Theta(R, Z, t) & P_z^Z(R, Z, t)
\end{bmatrix}. \tag{114}
\]

These can be computed explicitly using (109). The linear momentum balance of (59) becomes, with (75),

\[
\partial_A P_a^A + P_a^B H_{AB} - P_z^\Theta H_{\Theta z} F^\Theta_A + \left( P_a^A k \eta + \frac{\partial P_a^A}{\partial \eta} \right) \partial_A \eta = 0. \tag{115}
\]
Explicitly, this is the three partial differential equations
\[
\begin{align*}
\partial_R P^R_r + \frac{P^R_r}{R} - \gamma \frac{P^Z_r}{r} + \left( P^R_r k \eta + \frac{\partial P^R_r}{\partial \eta} \right) \partial_R \eta &= 0, \quad (116) \\
\partial_Z P^Z_\theta + \left( P^Z_k \eta + \frac{\partial P^Z_\theta}{\partial \eta} \right) \partial_Z \eta &= 0, \quad \partial_Z P^Z_z + \left( P^Z_k \eta + \frac{\partial P^Z_z}{\partial \eta} \right) \partial_Z \eta &= 0. \quad (117)
\end{align*}
\]

Now consider the kinetic equation for the internal state or order parameter. For
\[
\begin{align*}
D_K \to D_\Theta = l \eta, \quad \text{appropriately setting} \quad L_K \to L, \quad \text{and again using (75), the following}
\end{align*}
\]
reduced form of (64) is obtained:
\[
\dot{\eta} = -10^{-\beta} \frac{\dot{\epsilon}_0 l}{\Upsilon} \left[ Q - 2 \Upsilon \left( \partial_R \eta_R + \partial_Z \eta_Z \right) - 2 \Upsilon k \eta \left( \eta_R \partial_R \eta + \eta_Z \partial_Z \eta \right) + \psi (2 k \eta / l) \right], \quad (118)
\]
where \(Q\) is given by (110) and \(\psi\) by (104). This can be more conveniently written as
\[
\dot{\eta} = -10^{-\beta} \frac{\dot{\epsilon}_0 l}{\Upsilon} \left[ Q - 2 \Upsilon l \left( \partial_R \eta_R + \partial_Z \eta_Z \right) - 2 \Upsilon l k \eta \left( \eta_R \partial_R \eta + \eta_Z \partial_Z \eta \right) + 2 k \eta \psi \right],
\]
(119)
where terms in square brackets are of dimensions of energy per unit volume, \(L = 10^{-\beta} \dot{\epsilon}_0 l^3 / \Upsilon\), \(\dot{\epsilon}_0\) is a constant positive reference strain rate that will later be assigned as a mechanical boundary condition, and \(\beta\) is a parameter scaling the order parameter kinetics relative to the mechanical loading rate.

Trial general solutions of the system in (116), (117) and (119) are now considered. Momentum equations in (117) are trivially satisfied by the following conditions:
\[
\begin{align*}
\partial_Z P^Z_\theta &= \partial_Z P^Z_z = 0, \quad \partial_Z \eta = 0 \Rightarrow P^Z_\theta = P^Z_z (R, t), \quad (120) \\
P^Z_z &= P^Z_z (R, t), \quad \eta = \eta (R, t).
\end{align*}
\]
The kinetic equation (119) then reduces to
\[
\dot{\eta} = -10^{-\beta} \frac{\dot{\epsilon}_0 l}{\Upsilon} \left[ Q - 2 \Upsilon l \left( \partial_R \eta_R + \partial_Z \eta_Z \right) - 2 \Upsilon l k \eta \left( \eta_R \partial_R \eta + \eta_Z \partial_Z \eta \right) - \eta / R^2 \right] + 2 k \eta \psi,
\]
(121)
where terms in square brackets are of dimensions of energy per unit volume.

The system (120) and (121) essentially consists of two coupled nonlinear partial differential equations to be solved for the fields \(r(R, t) = R \hat{f}(R, t)\) and \(\eta(R, t)\). Physically, the radial displacement within the twisting cylinder and the microstructure-enabled torsional shearing are sought over some time period.

Particular solutions require specification of initial and boundary conditions. The former are simply those corresponding to null deformation and no microstructure development at the initial time:
\[
r(R, 0) = R, \quad \eta(R, 0) = 0. \quad (122)
\]
J. D. Clayton

Mechanical boundary conditions are essential boundary conditions for displacements in (10) for \( t \in [0, t_0] \) with \( t_0 \) the duration of the loading period:

\[
\begin{align*}
 r &= r(R_0, t) = \hat{f}(R_0) R_0 = \hat{f}_0 R_0 = r_0; \\
 \lambda(t) &= 1 \pm \hat{\epsilon}_0 t, \\
 \gamma(t) &= A \frac{\hat{\epsilon}_0 t}{L_0} = \Gamma(t)/L_0.
\end{align*}
\]

(123)

In the first of (123), the radial constraint is held fixed for all \( t \), where \( \hat{f}_0 = 1 \) for null expansion or contraction. In the second of (123), the imposed axial tension for \( \pm \to + \) (or compression for \( \pm \to - \)) increases linearly in magnitude with increasing time. In the third of (123), \( A \) is a dimensionless constant that relates imposed twisting and stretching rates, \( L_0 \) has already been introduced as the length of the cylinder, and \( \Gamma \) is the dimensionless cumulative shear strain. The usual continuum axisymmetric boundary condition \( r(0, t) = 0 \) is also prescribed such that the cylinder does not undergo cavitation along its centerline.

Numerical solution of the system (120) and (119) with initial and boundary conditions in (122) and (123) is straightforward. Time is discretized into increments \( \Delta t \), and the reference cylinder is subdivided into annular sections of radial increment \( \Delta R \) and volume \( 2 \pi L_0 \Delta R \). For each incremental time increase, the momentum balance in (116) is solved numerically leading to the updated radius \( r \) for a node corresponding to each radial increment. Then the kinetic law for the internal state, (121), is integrated explicitly to yield an updated value of \( \eta \) at each nodal location at the current time instant. First- and second-order finite differences are used to compute gradients of stress and internal state as needed. Time is updated and the same two equations are then solved again for the next instant. The procedure is iterated until the end time \( t_0 \) is reached.

4. Results and Analysis

Results for generic materials with physical properties typical of brittle or ductile polycrystalline solids are reported first in Sec. 4.1. These results serve to demonstrate general capabilities and physically correct trends predicted by the theory. Results for specific materials of interest, respectively boron carbide and ice, then follow in Secs. 4.2 and 4.3.

4.1. Representative nonlinear elastic, brittle, and ductile materials

Properties and parameters are listed in Table 1 for generic, isotropically elastic, solid materials. These are deemed physically representative of brittle polycrystals, such as typical ceramics, and ductile polycrystals, such as typical metals. Most properties such as elastic constants and surface energy as listed are self-explanatory. The elastic softening factor \( \zeta_0 \) is set to 0.1 for a ductile solid, implying a 10% loss of shear modulus at a material point \( X \) at its saturation limit when \( \eta(X) = 1 \). For a perfectly brittle solid, on the other hand, fracture results in complete loss of shear modulus at saturation since \( \zeta_0 = 1 \). The maximum inelastic (i.e. plastic)
shear strain accommodated by microstructure is the dimensionless product $\gamma_0 R_0$ for torsional loading. For a ductile material, a value of 0.5 is prescribed, while for a brittle material, less inelastic shearing occurs, such that a value of 0.05 applies. Since the direction of this isochoric inelastic deformation is purely that for twisting, and not extension/compression in axial and radial directions, the inelastic kinematic description physically corresponds to a plastically anisotropic material with a restricted set of glide or twin systems or one in which slip in certain directions is much easier than slip in other directions, for example molecular crystals with few system(s) [78], twinning in low-symmetry crystals [76], or basal slip in hexagonal crystals [79, 80]. A similar prescription of torsional plastic distortion was used in a dislocation-based model of deforming single crystal rods [43]. The dilatation from voids, point defects, or dislocations in a ductile material corresponds to a 1% maximum volume increase at saturation, giving a value of $\exp(k/2) = 1.01$ [3, 81]. In a brittle solid, crack opening contributes up to a maximum of 5% volume increase at local failure of the material, giving $\exp(k/2) = 1.05$ [82, 83]. The regularization length $l$ is taken as the cohesive process zone size [61, 84] for shear failure, similar to prior work [22, 26]:

$$l = \frac{4\mu_0 \Upsilon}{(1-\nu_0)\pi(\sigma_0)^2} = \frac{16\pi \Upsilon}{\mu_0(1-\nu_0)}. \quad (124)$$

Here $\sigma_0 = \mu_0/(2\pi)$ is the theoretical shear failure strength and $\nu_0 = (3K_0 - 2\mu_0)/(6K_0 + 2\mu_0)$ is Poisson’s ratio computed from the initial elastic constants $\mu_0$ and $K_0$ [3]. Finally, the kinetic factor $\beta$ is set to 1 to provide microstructure kinetics to occur at about the same rate as the mechanical loading for this example. The geometric (i.e. size) properties of the cylinder in its initial state are reported along with the loading protocols in Table 2. For axial extension, $\Lambda > 1$, while for axial compression, $\Lambda < 1$. During the loading process, the ratio $r_0/R_0$ is held fixed while $\Lambda$ and $\Gamma$ denoting respective axial and twisting deformations are incrementally updated.

Several average quantities are defined and studied in what follows. The total axial force $P(t)$ is the integral of the normal traction from $P_z^Z(R,t)$ over any cross-section of constant $Z$ with unit outward normal $N_Z = 1$:

$$P(t) = \pm \int_{-\pi}^{\pi} \int_0^{R_0} P_z^Z(R,t) N_Z R \, d\theta \, dR = \pm 2\pi \int_0^{R_0} P_z^Z(R,t) R \, dR. \quad (125)$$

Table 1. Parameters for generic ductile and brittle test materials.

<table>
<thead>
<tr>
<th>Property</th>
<th>Units</th>
<th>Value</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_0$</td>
<td>[GPa]</td>
<td>60</td>
<td>Shear modulus</td>
</tr>
<tr>
<td>$K_0$</td>
<td>[GPa]</td>
<td>100</td>
<td>Bulk modulus</td>
</tr>
<tr>
<td>$\Upsilon$</td>
<td>[J/m$^2$]</td>
<td>1.0</td>
<td>Surface energy</td>
</tr>
<tr>
<td>$\zeta_0$</td>
<td>0.1 or 1</td>
<td>Elastic softening factor for ductile or brittle fracture</td>
<td></td>
</tr>
<tr>
<td>$\gamma_0 R_0$</td>
<td>[rad]</td>
<td>0.5 or 0.05</td>
<td>Maximum inelastic shear strain for ductile or brittle material</td>
</tr>
<tr>
<td>$\exp(k/2)$</td>
<td>1.01 or 1.05</td>
<td>Weyl dilatation ratio for ductile or brittle fracture</td>
<td></td>
</tr>
<tr>
<td>$l$</td>
<td>[nm]</td>
<td>1.1</td>
<td>Regularization length from Eq. (124)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1 Ginzburg–Landau kinetic factor</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 2. Properties of cylindrical domain $\mathfrak{M}$ and loading parameters.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Value</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_0/l$</td>
<td>10</td>
<td>Cylinder radius</td>
</tr>
<tr>
<td>$L_0/l$</td>
<td>100</td>
<td>Cylinder length</td>
</tr>
<tr>
<td>$\dot{\varepsilon}_0$ [1/s]</td>
<td>1</td>
<td>Rate of applied axial straining in Eq. (123)</td>
</tr>
<tr>
<td>$A$</td>
<td>5</td>
<td>Ratio of twist rate to axial rate in Eq. (123)</td>
</tr>
<tr>
<td>$r_0/R_0$</td>
<td>0.94, 0.97, 1, 1.03, or 1.06</td>
<td>Radial constraint in Eq. (123)</td>
</tr>
<tr>
<td>$t_0$ [s]</td>
<td>1</td>
<td>Duration of loading</td>
</tr>
</tbody>
</table>

The sign of $P$ is chosen to correspond with that of axial loading in the second of (123). In subsequent figures, this force is normalized by the elastic axial stiffness $E_0A_0 = E_0\pi (R_0)^2$ with $E_0 = 9\mu_0 K_0/(\mu_0 + 3 K_0)$ the initial elastic modulus. The total twisting moment $M(t)$ results from the integral of the shear traction (force per unit area) due to $P_\theta^Z(R,t)/r$:

$$M(t) = \int_{-\pi}^{\pi} \int_0^{R_0} [P_\theta^Z(R,t)/r] r N_\theta R d\Theta dRdZ = 2\pi \int_0^{R_0} P_\theta^Z(R,t) R d\Theta dR. \quad (126)$$

In subsequent figures, this moment is normalized by the elastic torsional rigidity $I_0\mu_0 = \mu_0 \pi (R_0)^4/2$. The average value of the order parameter at time $t \in [0, t_0]$ is simply

$$\bar{\eta}(t) = \frac{1}{\pi (R_0)^2} \int_0^{L_0} \int_{-\pi}^{\pi} \int_0^{R_0} \eta(R,t) R d\Theta d\Omega dZ = \frac{2}{(R_0)^2} \int_0^{R_0} \eta(R,t) R dR. \quad (127)$$

The perfectly elastic solution is examined first to facilitate comparison with predictions of the complete theory that follow later. For perfect elasticity, $\eta(X,t) = 0$ for all $X \in \mathfrak{M}$ and for all $t \in [0, t_0]$. The axial force of (125) and twisting moment of (126) are shown in Figs. 1(a) and 1(b) for simultaneous axial compression and torsion. Analogous results for simultaneous axial extension and torsion are shown in Figs. 2(a) and 2(b). In both cases, the axial force and twisting moment always increase monotonically with increasing axial strain and increasing angle of twist. For compressive axial loading, the axial stress decreases with increasing $r_0/R_0$, while the twisting moment increases with increasing $r_0/R_0$. For tensile axial loading, the axial stress and the twisting moment both increase with increasing $r_0/R_0$. Results for twisting moment are not drastically different for tension versus compression. Results for axial force differ, on the other hand. The axial stiffness increases significantly for large compressive deformation, while the axial stiffness decreases slightly for large tensile deformation. The former trend is physically expected since most solids tend to demonstrate an increase in tangent bulk modulus under large compression. The logarithmic terms in the constitutive model for strain energy and stress in (105) and (109) enable this elastic nonlinearity.

Considered next are fully elastic–inelastic solutions obtained from the complete Finsler-geometric theory of Sec. 3. Results for the brittle material subjected to
Generalized pseudo-Finsler geometry

Fig. 1. Torsion and compression: nonlinear elastic solution for (a) compressive axial force $P$ versus stretch ratio $\Lambda$, (b) twisting moment $M$ versus angle of twist $\Gamma$.

Fig. 2. Torsion and extension: nonlinear elastic solution for (a) tensile axial force $P$ versus stretch ratio $\Lambda$, (b) twisting moment $M$ versus angle of twist $\Gamma$.

Simultaneous twisting and compression are shown in Fig. 3. Trends in compressive stress $P$ in Fig. 3(a) are similar to those for the elastic case in Fig. 1(a), with the tangent modulus increasing with increasing compression. On the other hand, the twisting moment shown in Fig. 3(b) demonstrates very different behavior than its elastic counterpart in Fig. 1(b), reaching a peak value then progressively relaxing with increasing applied angle of twist $\Gamma$. The moment $M$ is moderately affected by the radial constraint $r_0/R_0$. The reason for the relaxation is the increase in order parameter $\eta$ evident from the average $\bar{\eta}$ shown in Fig. 3(c) and the radial profiles shown in Fig. 3(d), the latter for $r_0/R_0 = 1$. As $\eta$ increases toward unity, the effective shear modulus $\mu_0(1 - \zeta_0\eta)^2$ decreases, leading to a loss of torsional rigidity. On the other hand, under compressive loading ($J^E < 1$), the tangent bulk modulus
Fig. 3. Brittle solid, torsion and compression: Finsler-geometric solution for (a) compressive axial force $P$ versus stretch ratio $\Lambda$, (b) twisting moment $M$ versus angle of twist $\Gamma$, (c) average order parameter $\bar{\eta}$ versus angle of twist $\Gamma$, (d) order parameter profile $\eta(R, t)$ for $r_0/R_0 = 1$.

does not decrease, such that the axial force $P$ is much less affected by increases in the order parameter. As is evident in Fig. 3(c), $\bar{\eta}$ increases most rapidly for larger values of $r_0/R_0$, since tensile radial loading promotes the inelastic dilatation associated with $\alpha^D$ and Weyl scaling of the metric due to $k > 0$. Furthermore, $\eta(R, t)$ tends toward larger local values with increasing $R/R_0$ since the effective shear driving force due to elastic shear strain increases with increasing radial distance from the axis of twist at $R = 0$.

Solutions for the brittle material subjected to simultaneous twisting and axial extension or stretching are shown in Fig. 4. Trends in tensile stress $P$ in Fig. 4(a) are very different from those for the elastic case in Fig. 2(a), with the tangent modulus and associated axial stress decreasing due to cumulative damage reflected by increases in the order parameter $\eta$. The twisting moment shown in Fig. 4(b) evolves similarly to the compressive case in Fig. 3(b), again reaching a peak value then progressively relaxing with increasing applied angle of twist $\Gamma$. Here, $M \to 0$ as
Fig. 4. Brittle solid, torsion and extension: Finsler-geometric solution for (a) tensile axial force $P$ versus stretch ratio $\Lambda$, (b) twisting moment $M$ versus angle of twist $\Gamma$, (c) average order parameter $\bar{\eta}$ versus angle of twist $\Gamma$, (d) order parameter profile $\eta(R,t)$ for $r_0/R_0 = 1$.

$\Gamma \to 0.5$, demonstrating lower shear strength under tensile loading than compressive loading for large angles of twist, in accordance with experiments on many materials [28, 29]. The moment $M$ is again moderately affected by the radial constraint $r_0/R_0$. Relaxation results from the increase in order parameter $\eta$ demonstrated by its average $\bar{\eta}$ in Fig. 4(c) and its radial profiles shown in Fig. 4(d), the latter again for $r_0/R_0 = 1$. As $\eta$ increases toward unity, the effective shear modulus $\mu_0(1 - \zeta_0\eta)^2$ decreases, leading to a loss of torsional rigidity. Furthermore, under tensile loading ($J^E > 1$), the tangent bulk modulus decreases in a similar fashion, such that the axial force $P$ is likewise strongly decreased by increases in the order parameter. Shown in Fig. 4(c), $\bar{\eta}$ increases most rapidly for larger values of $r_0/R_0$ for reasons discussed for Fig. 3(c), and it also increases more rapidly than the compressive loading case shown in the latter aforementioned figure. Local values of $\eta(R,t)$ increase with increasing $R/R_0$ in Fig. 4(d) as was the case for Fig. 3(d).
Fig. 5. Ductile solid, torsion and compression: Finsler-geometric solution for (a) compressive axial force $P$ versus stretch ratio $\Lambda$, (b) twisting moment $M$ versus angle of twist $\Gamma$, (c) average order parameter $\bar{\eta}$ versus angle of twist $\Gamma$, (d) order parameter profile $\eta(R, t)$ for $r_0/R_0 = 1$.

Results for the ductile material are now addressed. Those for simultaneous twisting and compression are shown in Fig. 5. The compressive stress $P$ of Fig. 5(a) behaves similarly to the nonlinear elastic case of Fig. 1(a), with the tangent modulus increasing with increasing compression. The twisting moment shown in Fig. 5(b), unlike its elastic counterpart in Fig. 1(b), attains maximal values then decreases slightly to a plateau as the angle of twist $\Gamma$ increases. The moment $M$ again is moderately affected by the radial constraint $r_0/R_0$. The decrease in moment relative to the perfectly elastic case again is due to increasing $\eta$ as is evident from the average $\bar{\eta}$ shown in Fig. 5(c) and the radial profiles in Fig. 5(d) for $r_0/R_0 = 1$. As $\eta$ increases toward unity, the effective shear modulus decreases, leading to a loss of torsional rigidity. Furthermore, for the ductile material with substantial $\gamma_0$, the plastic torsional strain accommodated by $\mathbf{F}^D$ further decreases the elastic strain and corresponding shear stress that contributes to $M$ in (126). As shown in Fig. 5(c), $\bar{\eta}$ increases with increasing $r_0/R_0$, for the aforementioned reason that tensile radial
loading promotes the inelastic dilatation associated with $\alpha^D$ and Weyl scaling of the metric. And for reasons noted already in the context of prior figures, $\eta(R, t)$ in Fig. 5(d) acquires larger values with increasing $R/R_0$.

Finally, solutions for the representative ductile material subjected to simultaneous twisting and axial extension or stretching are shown in Fig. 6. Trends in tensile stress $P$ in Fig. 6(a) are somewhat different from those for the elastic case in Fig. 2(a), with the tangent modulus and associated axial stress decreasing slightly according to modest increases in the order parameter $\eta$. The twisting moment shown in Fig. 6(b) evolves similarly to the compressive case in Fig. 5(b), again reaching a peak then decreasing essentially to a plateau applied angle of twist $\Gamma$. Twisting moment $M$ is again moderately affected by the radial constraint $r_0/R_0$. Comparison with Fig. 5(b) shows that torsional strength is predicted to be larger for compression than extension as is most evident at (large) $\Gamma \gtrsim 3$. The plateau is a product of the evolution of the order parameter $\eta$. The average value $\bar{\eta}$ of Fig. 6(c)
increases with increasing twist angle, as do local values in Fig. 6(d) for $r_0/R_0 = 1$.
Under tensile loading, the tangent elastic moduli decrease with increasing $\eta$, such that the axial force $P$ and moment $M$ are effectively decreased by increases in the order parameter. Here, since $\zeta_0 = 0.1$ for a ductile solid, this decrease in moduli is rather slight. The average shown in Fig. 6(c) and local values shown in Fig. 6(d) tend to increase with increasing $r_0/R_0$ and $R/R_0$, respectively, in accordance with trends and reasons reported in the context of prior figures.

Several additional phenomena are evident upon comparison of results for the brittle solid in Figs. 3 and 4 with those for the ductile solid in Figs. 5 and 6. Under the same initial and boundary conditions, local and global average values of the order parameter $\eta(R,t)$ tend to increase more rapidly for the former case, i.e. the brittle solid. This leads to attainment of peak twisting moment $M$ of lower magnitude at a smaller applied angle of twist $\Gamma$ in the brittle material compared to the ductile material. For tensile loading, the peak axial force supported by the brittle material in Fig. 4(a) is also much lower than the axial load supported by the ductile material, with the latter in Fig. 6(a) never attaining a local maximum. These relative trends are physically representative of classical brittle and ductile solids, where the former brittle materials tend to demonstrate relatively less inelastic deformation and more abrupt load decreases upon fracture. The latter (ductile) materials tend to demonstrate a slower rate of softening, larger applied deformation at peak or maximal loads, and maintenance of some residual strength under finite plastic deformations.

Ideally, physical meaning should be attached to the state variable $D$ or order parameter $\eta = D/l$, and the variable should be measurable through experiments or atomistic simulations. For the present application to combined torsion and extension or compression, inelastic deformation manifests as shearing via (103) and possible dilatation via (101). First consider a brittle solid. Inelastic deformation is usually manifested by sliding and/or opening of micro-cracks. The state variable associated with microstructure ($D$ or $\eta$) can in this case be linked to the crack density and crack opening displacement through a suitable homogenization procedure that relates inelastic deformation and modulus degradation to the arrangement of cracks within an element of material [82, 85, 86]. Though tedious, experimental characterization of micro-crack distributions in brittle solids is possible [85], which would enable direct validation of the present theory when applied to a specific material system for which such experimental data exist. Now consider a ductile solid. Inelastic deformation takes place via glide of dislocations, generally small dilatation may arise from nonlinear elastic and dislocation core effects [87], and more severe expansion may occur in conjunction with void growth leading to ductile rupture. Presuming that a physically motivated equation relating dislocation density to cumulative plastic deformation [88, 89] exists for the present applications to monotonic loading, then the order parameter can be directly associated with dislocation density. Furthermore, the dilatation from voids can be directly linked to their number and size distributions [85]. Standard procedures exist for counting...
dislocation lines \cite{90} and measuring void characteristics in sectioned solids \cite{85}. Thus, evolution of the internal variable predicted by the model could be validated by comparison with dislocation and void data for specific metallic systems. Further insight may be obtained by consideration of general principles for assigning physically identifiable and experimentally measurable internal variables, and appropriate length scales for regularization, for gradient theory as set forth in \cite{91,92}.

### 4.2. Results and analysis: Boron carbide crystals

Boron carbide is a stiff crystalline ceramic used in industrial applications requiring strong, hard materials. Under severe deformation, for example occurring in impact or indentation loading, boron carbide may undergo a solid–solid phase change from crystalline (rhombohedral or trigonal symmetry) to amorphous or glassy phases \cite{52,93–95}. Prior work has applied a theory similar to the present one to study this material subjected to static tension and compression \cite{22}, simple shear \cite{23}, and shock compression \cite{26}. In this paper, the theory is newly applied to study torsional loading of a small single crystal, with and without superimposed axial compression.

Properties for boron carbide are listed in Table 3 with supporting references or remarks. The order parameter physically reflects the change of phase from fully crystalline ($\eta = 0$) to glass ($\eta = 1$), whereby a loss of shear strength is demonstrated by the latter phase \cite{22,95}. The density increases by approximately 4% during the structural transformation \cite{53,94}. The elastic strength degradation and density increase are reflected in the model by the prescribed values of $\zeta_0$ and $k$ in Table 3. Specifically with regard to the loading protocol, the planes of twisting $Z=\text{constant}$ correspond to planes in the crystal prone to amorphous band formation, e.g. (01\bar{1}1) in hexagonal Miller indices, or to others listed in \cite{97,98} that do not undergo deformation twinning. Inelastic shear $\gamma_0$ correlates with the behavior observed for simple shearing in atomic simulations \cite{98} or six-fold symmetric shearing on pyramidal planes for shock compression along the \textit{c}-axis \cite{26}. Elastic anisotropy is omitted.

Loading of a cylinder of dimensions identical to those in Table 2 is assigned according to the same initial and boundary conditions given by (122) and (123). Since the kinetic coefficient $L$ is scaled by the applied strain rate $\dot{\epsilon}_0$, the kinetic

<table>
<thead>
<tr>
<th>Property [Units]</th>
<th>Value</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_0$ [GPa]</td>
<td>193</td>
<td>Shear modulus \cite{94}</td>
</tr>
<tr>
<td>$K_0$ [GPa]</td>
<td>237</td>
<td>Bulk modulus \cite{94}</td>
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<tr>
<td>$\Sigma$ [J/m$^2$]</td>
<td>3.27</td>
<td>Surface energy \cite{20,22,26}</td>
</tr>
<tr>
<td>$\zeta_0$</td>
<td>1</td>
<td>Elastic softening factor for brittle fracture \cite{22,23}</td>
</tr>
<tr>
<td>$\gamma_0 R_0$ [rad]</td>
<td>0.5</td>
<td>Maximum inelastic shear strain at localization and rupture \cite{20,95}</td>
</tr>
<tr>
<td>$\exp(k/2)$</td>
<td>0.96</td>
<td>Weyl densification ratio \cite{22,53,95}</td>
</tr>
<tr>
<td>$l$ [nm]</td>
<td>0.97</td>
<td>Regularization length from Eq. \cite{124} or \cite{20}</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1</td>
<td>Ginzburg–Landau kinetic factor (generic brittle solid of Table 1)</td>
</tr>
</tbody>
</table>

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coefficient $\beta$, loading rate, and time duration $t_0$ do not affect the present results and are left unchanged from their values considered in Sec. 4.1.

Results for combined torsional and finite compressive loading are shown in Fig. 7. Recall that $r_0/R_0$ is held fixed while the axial compression $\Lambda$ and twist angle $\Gamma$ are updated incrementally. The normalized axial force $P$ of Eq. (125), positive in compression, is given in Fig. 7(a). Compressive force increases with increasing radial constraint, i.e. with decreasing $r_0/R_0$. An increase in the tangent modulus is evident at large compression commensurate with the logarithmic contributions to the strain energy function resulting in an increased bulk modulus at large compression.

The normalized twisting moment $M$ of Eq. (126) is shown in Fig. 7(b). Its values tend to reach local maxima at $\Gamma \approx 0.5$ then subsequently decrease with increasing angle of twist. This trend results from the shear accommodation from the inelastic strain as well as the decrease in tangent shear modulus with increasing transformation.
fraction reflected by $\eta > 0$. The peak moment varies on the order of 10% depending on the choice of $r_0/R_0$. The average order parameter for the cylinder computed via (122) is reported in Fig. 7(c). Behavior does not vary substantially with the radial constraint. Transformation kinetics are driven by the combination of modulus degradation, which is promoted by tension, and by inelastic densification, which is promoted by compression. Local profiles for $r_0/R_0 = 1$ are given in Fig. 7(d).

Elastic shear strains and shear stress increase with increasing $R$ under torsion, leading to increasing $\eta$ with increasing distance from the centerline of the cylinder.

Comparison with the torsion results of Fig. 7 with those for pure axial compression proves informative. The latter are shown in Fig. 8. For pure compression, parameter $A = 0$ in (123), and inelastic shear constant $\gamma_0 = 0$ since there is no preferred direction for transformation shearing. Micro-deformations associated with $\eta$ can be considered randomly oriented in planes of constant $Z$ such that the net inelastic shearing deformation is zero. Inelastic volume changes are still captured by $\alpha^D$ and the Weyl scaling factor $k$ which remain unchanged in respective form and value. Shown in Fig. 8(a) is the normalized compressive axial force $P$, while shown in Fig. 8(b) is the average order parameter representative of the phase transformation. The twisting moment $M = 0$ for pure compression, and profiles of $\eta(R, t)$ do not vary with $R$, such that $\eta(R, t) = \bar{\eta}(t), \forall R \in [0, R_0]$. Comparing results of Fig. 8(a) with those of Fig. 7(a), compressive stress is slightly larger when twisting is not imposed, especially for moderate $\Gamma$. Comparing results of Fig. 8(b) with those of Fig. 7(c), the average order parameter increases much more slowly with increasing compressive strain when twisting is not imposed. In other words, twisting and the associated shearing stress tend to promote rapid transformation and softening when superposed with finite compression.

Finally considered are effects of variable compression superposed with finite torsional deformation, all for perfect radial constraint. For these solutions, the essential
Fig. 9. Boron carbide, torsion and null to large compression: Finsler-geometric solution for (a) compressive axial force $P$ versus angle of twist $\Gamma$, (b) twisting moment $M$ versus angle of twist $\Gamma$, (c) average order parameter $\bar{\eta}$ versus angle of twist $\Gamma$, (d) order parameter profile $\eta(R,t)$ for $\Lambda_0 = 1$.

Newly introduced is the constant $\Lambda_0 \in [0.5, 1.0]$ that produces final compression ratios $\Lambda(t_0) = \Lambda_0$, noting that $\Lambda_0 = 1$ corresponds to pure torsion. Reported in Fig. 9(a) is the normalized axial force of (125), positive in compression. The compressive force increases with decreasing $\Lambda_0$, i.e. with increasing compressive strain, as physically expected. For the case with $\Lambda_0 = 0$, tensile axial stress is observed at small to moderate $\Gamma$. This phenomenon is a result of the densification that accompanies the structural transformation with increasing values of $\eta$, leading to a compensating tensile volumetric elastic deformation and tensile pressure. Similar trends have been reported in atomic simulations [98] and continuum simulations of...
simple shear \[23\], with tensile pressure eventually contributing to cavity formation and rupture in the former. The twisting moment \(M\) in Fig. 7(b) demonstrates similar behavior to that reported in Fig. 3(b) for \(r_0/R_0 = 1\), though when \(\Lambda_0\) increases, the decrease in \(M\) at large twist is reduced relative to the case in Fig. 7(b). Likewise, the average transformation variable \(\tilde{\eta}\) in Fig. 7(c) behaves very much like the results in Fig. 7(c) for \(\Gamma \lesssim 2.5\), irrespective of \(\Lambda_0\). At larger \(\Gamma\), the transformation rate is accelerated by increasing axial compression, i.e. by a decrease in \(\Lambda_0\). Comparison with Fig. 8(b) reveals a marked decrease in the transformation rate when mechanical shearing from torsion is removed. Local values of \(\eta(R,t)\) differ slightly in Figs. 9(d) and 7(d), with finite compression accelerating the transformation kinetics at smaller \(R/R_0\).

Summarizing the comparisons of Figs. 7–9, superposition of small to moderate axial compression (\(\Lambda_0 > 0.75\)) on finite torsional loading results in very modest effects on overall torsional strength and transformation behavior. Superposition of finite torsion on compression, the latter whether small or large, results in drastic changes to torsional strength and microstructure evolution. Thus, one may conclude that the shearing response dominates the compressive response with regard to large inelastic deformation and softening in boron carbide. This conclusion agrees with observations from atomic simulations wherein instability and structure collapse were found to occur much more readily when shearing was superposed with compression along the \(c\)-axis \[96, 100\].

The state variable in the present application (\(D\) or, in dimensionless form, \(\eta\)) is directly associated with the volume fraction of transformed substance (crystal to glass) within a local element of material. The size and shape of amorphous zones in deformed or fractured boron carbide specimens have been measured using Raman spectroscopy and transmission electron microscopy in static compression and indentation tests \[53, 93\]. No high-pressure torsion data exist for single crystals of this material, however, so direct comparison and validation of the present predictions await such experimental measurements.

### 4.3. Results and analysis: Ice crystals

The final application of the theory and method of solution is directed toward single crystals of ice, i.e. frozen water. The intent is qualitative validation of the predictive capabilities of the model by comparison with torsion experiments conducted on ice single crystals maintained in an environment of approximately \(-15\)°C \[51\]. Cylinders were oriented for torsion about the \(c\)-axis with twisting occurring by relative rotations of basal \(\{0001\}\) planes. In these creep-type experiments, a constant torque was applied and the shear strain or rotational deformation was recorded over a long time period, e.g. on the order of \(3 \times 10^4\) s. Diameters of the specimens varied from 20 mm to 40 mm, with lengths of the cylinders ranging from 50 mm to 60 mm. Maximum shear strains obtained, presumably prior to fracture, rupture, shear localization, and/or melting, ranged from 0.04 to 0.07. Dislocation content was measured...
Table 4. Parameters for ice (hexagonal H$_2$O).

<table>
<thead>
<tr>
<th>Property [Units]</th>
<th>Value</th>
<th>Remarks</th>
</tr>
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<tbody>
<tr>
<td>$\mu_0$ [MPa]</td>
<td>3520 Shear modulus</td>
<td>[102]</td>
</tr>
<tr>
<td>$K_0$ [MPa]</td>
<td>8900 Bulk modulus</td>
<td>[102]</td>
</tr>
<tr>
<td>$\Upsilon$ [MPa-mm]</td>
<td>0.4 Surface energy (see text)</td>
<td></td>
</tr>
<tr>
<td>$\zeta_0$ [rad]</td>
<td>0.1 Elastic softening factor for ductile fracture (see Sec. [14])</td>
<td></td>
</tr>
<tr>
<td>$\gamma_0 R_0$ [rad]</td>
<td>0.05 Inelastic shear strain at pending localization and rupture [51]</td>
<td></td>
</tr>
<tr>
<td>$\exp(k/2)$</td>
<td>0.991 Weyl densification ratio from Eq. (129)</td>
<td></td>
</tr>
<tr>
<td>$l$ [mm]</td>
<td>1.0 Regularization length from approximate slip band width [80]</td>
<td></td>
</tr>
<tr>
<td>$\rho_0/\rho_1$</td>
<td>0.92 Density ratio of solid ice to water [102]</td>
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<td>$\beta$</td>
<td>1 Ginzburg–Landau kinetic factor (generic ductile solid in Table [4])</td>
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using X-ray diffraction. The main results of the study were variations in cumulative strain, and an increase in dislocation density and strain gradient, with decreasing sample diameter. The torsional strain was accommodated by screw dislocations on the basal plane.

Properties for ice are provided in Table 4. Note that natural ice crystallizes in a hexagonal structure. With regard to the theory, the order parameter now physically reflects the accumulation of micro-deformation attributed to basal slip of dislocations, the dominant mode of plastic flow in solid ice [80]. A (nearly) perfect crystal without (significant) dislocation content or structural disorder is denoted by the initial condition $\eta = 0$. Slip attains a saturation limit as $\eta \rightarrow 1$, at which point partial melting associated with disorder occurs. A slight loss of elastic shear strength accompanies this microstructure evolution process, with $\zeta_0 = 0.1$ corresponding to a ductile material as studied in Sec. 4.1. Accordingly, at saturation, partial amorphization and/or melting related to the solid to liquid transition occurs. Denoting the density of natural ice by $\rho_0$ and the density of water by $\rho_1$, the value of $k$ then corresponds to an approximate densification of 1% at saturation:

$$\exp\left(\frac{k}{2}\right) = \frac{\rho_0}{\zeta_0 \rho_1 + (1 - \zeta_0) \rho_0}.$$ (129)

Volume decreases with increasing disorder leading to a negative $k$ since ice crystals are less dense than the amorphous or liquid phase. Inelastic shear $\gamma_0$ is taken as the plastic deformation measured at the conclusion of the experiments. The surface energy corresponds to formation of a two-sided zone of pending failure by localization or fracture [80], where the work of the maximum experimental shear stress $\tau_0 = 0.2$ MPa at $R_0$ [51] acts over a distance of $2l$ for formation a two-sided surface, i.e. $\Upsilon = 2\tau_0 l$. Elastic anisotropy, moderate in ice [80], is again omitted.

Cylinders of dimensions shown in Table 5 are assigned initial conditions according to [122]. Boundary conditions correspond to pure torsion rather than those of [123]. Deformation-controlled torsion is imposed via

$$r = r(R_0, t) = r_0 = R_0; \quad \Lambda = 1, \quad \gamma(t) = \frac{A}{L_0} \eta_0 t = \frac{\Gamma(t)}{L_0}.$$ (130)
Table 5. Properties of cylindrical domain $\mathcal{M}$ and loading parameters.

<table>
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<tr>
<th>Quantity [Units]</th>
<th>Value</th>
<th>Remarks</th>
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<tbody>
<tr>
<td>$R_0/l$</td>
<td>10, 12, 15</td>
<td>Cylinder radius [S1]</td>
</tr>
<tr>
<td>$L_0/l$</td>
<td>60</td>
<td>Cylinder length [S1]</td>
</tr>
<tr>
<td>$\dot{\epsilon}_0$ [1/s]</td>
<td>$1 \times 10^{-5}$</td>
<td>Total effective rate of twist in Eq. (130)</td>
</tr>
<tr>
<td>$\tau_0/R_0$</td>
<td>1</td>
<td>Radial constraint in Eq. (130)</td>
</tr>
<tr>
<td>$t_0$ [s]</td>
<td>$2 \times 10^4$</td>
<td>Duration of loading [S1]</td>
</tr>
</tbody>
</table>

The value of $A = 0.2$ in Table 5 produces a maximum shear strain at $R = R_0$ and $t = t_0$ on the order of that observed in the experiments. The applied strain rate $\dot{\epsilon}_0$ and end time $t_0$ are consistent with those from the torsion experiments [S1]. Parameter $l$ is prescribed a value on the order of the width of slip or shear bands observed in compression experiments on ice [S0]. Use of a regularization length on the order of atomic spacing would be prohibitive for computational reasons since discretization must be fine enough to resolve this length [S1]. The kinetic coefficient $\beta$ is maintained from Sec. 4.1.

Results for the mechanical response of three cylinders with radii listed in Table 5 are reported in Fig. 10, obtained via the numerical method outlined at the end of Sec. 3.2. The average shear stress $\bar{\tau}$ is shown in Fig. 10(a), computed via

$$\bar{\tau}(t) = \frac{M(t)}{2\pi(R_0)^2L_0},$$

with the torque $M$ found from (126). This is the stress that would be required on the lateral surface to maintain the torque shown in Fig. 10(b) and its definition is consistent with the experimentally applied shear stress $\tau_0$ reported in [S1]. The experimental data in Fig. 10(a) correspond to the average stress $\tau_0 = 0.2$ MPa applied at a nearly constant, time-independent value in the creep tests for all three cylinders. Model predictions in Fig. 10(a) demonstrate a period of nearly constant average shear stress with increasing $\Gamma$, with values close to the experimental data. The period of constant stress is preceded for each size of cylinder by elastic loading to a local peak value, then a drop in shear stress to the steady-state value with increasing $\Gamma$. The peak stress demonstrates a size effect, with increasing values of local maxima of $\bar{\tau}$ accompanying increasing cylinder radius $R_0$. The steady-state value at larger $\Gamma \gtrsim 0.05$ is independent of the size of the cylinder, a result consistent with the experimental test protocol [S1].

The normalized twisting moment $M$ is reported in Fig. 10(b). Results demonstrate a notable size effect, whereby the smaller the cylinder radius, the larger the normalized torsional strength. This “smaller is stronger” effect is consistent with physical observations on other materials deformed in torsion [S4]. The experimental creep data for ice showed variability in cumulative strain and strain rate with sample size, but a demonstration of a trend of increasing strength with decreasing cylinder radius was inconclusive from the limited number of tests reported [S1]. Here in the present modeling context, as in these experiments, no attempt is made to correlate
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Fig. 10. Ice: Finsler-geometric solution for (a) average shear stress at \( R_0 \) versus angle of twist \( \Gamma \), (b) twisting moment \( M \) versus angle of twist \( \Gamma \), (c) average order parameter \( \bar{\eta} \) versus angle of twist \( \Gamma \), (d) order parameter profile \( \eta(R, t) \) for \( R_0/L_0 = 1/5 \).

strength with dislocation density as has been successfully accomplished elsewhere via use of gradient plasticity theory [34]. A dependence of yield on cylinder radius was also found in another somewhat more recent application of continuum dislocation theory [43]. Polycrystalline ice demonstrates Hall–Petch strengthening [80], i.e. increasing strength with decreasing grain size [103]. Note that the normalized moment computed by the method reported for Fig. 10(b) is independent of size ratio \( R_0/L_0 \) for a purely elastic process with \( \eta(R, t) = 0 \) everywhere.

The average order parameter \( \bar{\eta} \), physically representative of cumulative dislocation motion and modest structural changes associated with defect accumulation, is shown in Fig. 10(c). Values increase slightly with increasing cylinder radius, perhaps since local driving shear stresses increase with increasing \( R \). Profiles of the local value of \( \eta(R, t) \) were very similar for all three values of \( R_0 \); results for the intermediate case of \( R_0/L_0 = 1/5 \) are given in Fig. 10(d). Similarly to trends reported for the
torsion of generic materials in Sec. 4.2 and boron carbide in Sec. 4.3, transformation occurs most rapidly as the outer radius of the cylinder is approached, presumably a result of the increasing driving force mentioned above. Interestingly, as \( R \to 0 \), the order parameter \( \eta(R, t) \to 0 \), implying negligible driving force near the axis of rotation. This trend is observed in boron carbide under pure torsion in Fig. 9(d) but is absent in prior results such as those of Fig. 7(d) where compression and torque are applied simultaneously.

In the present application, structure variable \( \eta = D/l \) can be physically related to dislocation content via introduction of a supplementary equation relating dislocation density (length per unit reference volume), denoted here as \( \rho_D \), and inelastic shear (angle of twist per unit length) \( \gamma^D(\eta) \) of (103). For the particular slip system geometry considered here and in corresponding experiments on ice single crystals, the appropriate relation is given in [51] as

\[
\rho_D(\eta) = \frac{\gamma^D(\eta)}{b}, \tag{132}
\]

where \( b \) is the magnitude of the Burgers vector of screw dislocations on the basal plane with a value of 0.452\,nm [80]. The predicted dislocation density \( \rho_D \) at \( \eta = 1 \) corresponds exactly to that which would be reported from the experiments at \( \gamma^D = \gamma_0 \), with values on the order of \( 10^{10} \, \text{m}^{-2} \) obtained at the termination of each torsion test under the perfectly plastic assumption of \( \gamma \approx \gamma^D \). Accuracy of (132) was confirmed by X-ray diffraction [51]. However, \( \textit{in situ} \) measurements of dislocation density throughout the time history deformation, not available from [51], are needed to validate predicted evolution of the state variable during earlier stages of the deformation process, which is why quantitative comparisons are not given here.

5. Conclusions

A continuum physics theory combining geometric elements from generalized pseudo-Finsler geometry with thermodynamic aspects of phase field theory has been developed and applied to describe the mechanics of torsional deformation. The general tensor formulation of the dynamic theory has been presented, followed by new derivation of the governing nonlinear partial differential equations pertaining to torsional loading with possible axial extension/compression and radial constraint. Numerical solutions have been obtained for representative generic brittle and ductile solids, followed by solutions for boron carbide and ice single crystals. Trends in results agree with physical observations. Results for boron carbide demonstrate the primary importance of shear on evolution of structural transformation and failure, with compression a secondary factor. Results for ice demonstrate periods of steady-state average shear stress, independent of cylinder radius, that closely align with experimental values. Furthermore, the overall normalized torsional resistance is shown to increase with decreasing cylinder size, a phenomenon that has been observed for other elastic–plastic crystalline solids.
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References


Generalized pseudo-Finsler geometry


Generalized pseudo-Finsler geometry


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